Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017-OA-0078

Vol. **11**, No. 3, pp. 655-672
August 2018

A Weak Galerkin Finite Element Method for Elliptic Interface Problems with Polynomial Reduction

Bhupen Deka[1,*]

[1] *Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati 781039, India*

Received 27 June 2017; Accepted (in revised version) 14 December 2017

Abstract. This paper is concerned with numerical approximation of elliptic interface problems via week Galerkin (WG) finite element method. This method allows the usage of totally discontinuous functions in approximation space and preserves the energy conservation law. In the implementation, the weak partial derivatives and the weak functions are approximated by polynomials with various degrees of freedom. The accuracy and the computational complexity of the corresponding WG scheme is significantly impacted by the selection of such polynomials. This paper presents an optimal combination for the polynomial spaces that minimize the number of unknowns in the numerical scheme without compromising the accuracy of the numerical approximation. Moreover, the new WG algorithm allows the use of finite element partitions consisting of general polytopal meshes and can be easily generalized to high orders. Optimal order error estimates in both H^1 and L^2 norms are established for the present WG finite element solutions.

AMS subject classifications: 65N15, 65N30

Key words: Elliptic, interface, finite element method, weak Galerkin method, optimal error estimates, low regularity.

1. Introduction

The physical world is replete with examples of free surfaces, material interface and moving boundaries. It is the case when two distinct materials or fluids with different conductivities or densities or diffusions are involved. Mathematical modeling of such problems often lead to differential equations with discontinuous coefficients and singular sources. This class of problems is commonly called interface problems. Elliptic interface problems have a variety of applications in many scientific and engineering disciplines, including fluid dynamics [23], materials science [42] and biological systems [10].

Owing to its mathematical complexity and essential importance in a number of application areas, the study of interface problems has evolved into a well defined field in applied and computational mathematics. The past few decades have witnessed intensive research

*Corresponding author. *Email address:* bdeka@iitg.ernet.in (B. Deka)

activity in interface problems. Finite element method (FEM) is an another class of important approaches for interface problems and a wide variety of FEM approaches have been proposed in the literature. There are two major classes of FEM depending on the choice of the discretization, namely, interface-fitted FEMs and unfitted FEMs. In fitted FEMs, the discretization is made in such a way that the grid line is either follow the actual interface or an approximation of the smooth interface. In unfitted FEMs, the discretization is independent of the location of the interface. We first give a brief account of the development of the finite element methods for elliptic interface problems. One of the first finite element methods treating elliptic interface problem has been studied by Babuška in [3]. Since then, elliptic interface problems have attracted extensive effort in the FEM community as well (e.g., [5,6,8,9,12,14,16]; and reference therein for fitted FEM). The numerical solution for elliptic interface problems by means of unfitted finite element methods are investigated by several authors in [4,5,19,25,30]. However, in this type of methods, mesh generation and refinement can be a technically demanding and computationally time consuming process. To avoid the complicated mesh generation process, immersed FEMs have been proposed to allow the interface to cut through elements so that simple structured Cartesian meshes can be employed. This renders immersed FEMs great popularity in solving a variety of interface problems, such as elliptic interface problem [13, 17, 21, 24, 28], elasticity interface problems [18], to name only a few. In fact, immersed FEMs can be regarded as the Galerkin formulations of finite difference based interface schemes. It is not surprised that key ideas of many immersed FEMs actually come from the corresponding finite-difference based interface schemes. For a more detailed discussion on finite difference methods for interface problem, we refer to [2, 20, 27]. Rigorous convergence analysis of most other finite difference based elliptic interface schemes is not available yet. In general, it is quite difficult to analyze the convergence of finite difference based interface schemes because conventional techniques used in Galerkin formulations are not applicable for collocation schemes.

The objective of the present work is to propose and analyze weak Galerkin finite element method for elliptic interface problems. The weak Galerkin finite element methods (WG-FEM for short) introduced by [38] refers to the numerical techniques for partial differential equations where the differential operators (e.g., gradient, divergence, curl, Laplacian) are approximated by weak forms. Like the DG methods, WG-FEM makes use of discontinuous functions in the finite element procedure which endows WG-FEM with high flexibility to deal with geometric complexities and boundary conditions. Unlike DG methods, WG-FEM enforces only weak continuity of variables naturally through well defined discrete differential operators. Therefore, weak Galerkin methods avoid pending parameters resulted from the excessive flexibility given to individual elements. As a consequence, WG-FEM are absolutely stable once properly constructed. In [38], a weak Galerkin method was introduced and analyzed for second order elliptic equations based on a *discrete weak gradient* arising from local *RT* ([37]) or *BDM* ([7]) elements. Due to the use of the *RT* and *BDM* elements, the weak Galerkin finite element formulation of [38] was limited to classical finite element partitions of triangles ($d = 2$) and tetrahedra ($d = 3$). A computational study of the weak Galerkin method for second-order elliptic equations has been carried out