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A Splitting Collocation Method for Elliptic Interface Problems

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Abstract. An effective method is proposed for interface problems by combining the domain decomposition method and the collocation method. The main idea is to split the whole domain into two non-overlapped sub-domains and enforce the interface conditions to obtain two sub-problems. Each sub-problem is solved by the collocation method. And a simple iterative algorithm is presented to achieve the jump conditions at the interface. The method provided can be used to solve interface problems with both linear and a special type of non-linear jump conditions. Numerical experiments reveal that our method is fourth-order accurate for interface problems.

AMS subject classifications: 65M50, 65L50, 65N50

Key words: Non-overlapping domain decomposition, elliptic interface problem, moving collocation method.

1. Introduction

Differential equation with discontinuous coefficients or singular source terms is one of the most important mathematical models in many applications. Examples include moving interface problem, composite material, crystal growth, etc. This kind of differential equation has attracted considerable attention from scientists and engineers. A lot of efficient numerical methods have been developed in the last three decades. One of the most important methods is the immersed boundary method (IBM) which was proposed in the pioneering work of Peskin [1]. By using a discretized delta-function, the method leads to the first-order accuracy. Leveque and his co-workers developed the immersed interface method

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(IIM) which has achieved much success [2,3]. The main idea of IIM is to incorporate the jump conditions into the numerical scheme to achieve a second-order accuracy.

High-order numerical schemes have also been developed [4–7]. In [4], a high-order immersed interface method was proposed by using a wider set of grid stencils. The matched interface and boundary (MIB) approach [5,6] made use of fictitious domains. Then the standard high-order central finite difference (FD) method was applied across the interface without the loss of accuracy. These methods can achieve any high-order accuracy in principle, and have been well studied recently [8–12]. Other numerical methods for elliptic interface problems are also developed including the interface finite volume method [13], the non-symmetric positive definite finite element method [14], etc.

Recently, Loubenets and his co-workers [15] proposed a fourth-order accurate finite element method for one-dimensional interface problems by modifying the basis functions in the vicinity of the interface. In [16], a spectral collocation and radial basis function methods were developed for one-dimensional interface problems which attained very high accurate results. However, these methods are only proposed for linear interface problems.

Consider the following elliptic interface problem,

$$
\begin{aligned}
&-\left(\beta u_x\right)_x + ru = f(x), \quad x \in (x_l,\alpha)\cup(\alpha,x_r), && \text{(1.1a)}\\
&u(x_l) = u_{b^-}, \qquad\qquad\quad u(x_r) = u_{b^+}, && \text{(1.1b)}\\
&[u]_\alpha = g(x,u(\alpha-),u(\alpha+)), && \text{(1.1c)}\\
&[\beta u_x]_\alpha = h(x,u(\alpha-)), && \text{(1.1d)}
\end{aligned}
$$

where α is the location of the interface, $[u]_\alpha$ and $[\beta u_x]_\alpha$ are the jumps at the interface defined by

$$
\begin{aligned}
[u]_\alpha &= \lim_{x\to\alpha+0} u(x) - \lim_{x\to\alpha-0} u(x),\\
[\beta u_x]_\alpha &= \lim_{x\to\alpha+0} \beta u_x(x) - \lim_{x\to\alpha-0} \beta u_x(x).
\end{aligned}
$$

The functions $\beta(x)$ and $r(x)$ are smooth on both $\Omega^- = (x_l,\alpha)$ and $\Omega^+ = (\alpha,x_r)$, and $f \in L^2(\Omega)$. This problem is a very general one including equations with discontinuous coefficients or singular source terms. Moreover, the jump conditions can be nonlinear which is not considered in [15,16].

We propose a domain decomposition collocation method to solve this general interface problem effectively. Inspired by the idea of [17] where some domain decomposition methods were derived to solve the poisson problem, we split the domain into two sub-domains at the interface. Then the interface conditions are enforced on two artificial boundaries. By doing this, the problem is divided into two sub-problems. Each sub-problem is solved by the collocation method. Then a very simple iterative procedure is presented by alternately solving these two sub-problems.

Similar idea of sub-domain method was presented for one dimensional interface problems [18]. In their method, interface conditions were relaxed together with the internal equations, and this led to an iterative method on the whole set of grid values. To construct the finite difference numerical schemes near the interface, some ghost points were used