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Dissipativity of θ-Methods for Nonlinear Delay Differential Equations

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Abstract. This paper concerns dissipativity of one-leg θ-methods and linear θ-methods for nonlinear delay differential equations (DDEs). Firstly, we obtain the absorbing set generated by the numerical methods and then prove that the methods can inherit the dissipativity of underlying system. It is shown that the radius of the absorbing set generated by the discrete system goes to that generated by the continuous system as the step size goes to zero. The estimate of the radius is sharp in this sense. Secondly, because the model considered is a very broad class of differential equations with or without delays, the main results obtained provide a unified treatment for ordinary differential equations (ODEs) and DDEs with constant delays and variable delays (including bounded and unbounded variable delays). In particular, the results are also new even in the case of ODEs. Finally, numerical experiments are given to support our theoretical results.

AMS subject classifications: 65L05, 65L20

Key words: Dissipativity, delay differential equation, one-leg θ-method, linear θ-method.

1. Introduction

In many applications there often occur systems which have every one of their solutions driven into fixed bounded domain and kept there under further increase of time. Such systems are called dissipative ones. Solutions of dissipative systems are called limit (finally) bounded.

Since exact solutions are rarely known, one needs approximations of solutions of such dissipative systems. In the study of numerical methods, it is natural to ask whether those discrete systems preserve the dissipativity of the continuous systems.

Since the 1990s considerable process has been made in disspativity analysis of numerical methods. The papers [5, 6, 10, 11, 21] focus on the numerical methods for ordinary differential equations (ODEs). For the delay differential equations (DDEs) with constant delay, sufficient conditions for the dissipativity of analytical and numerical solutions are presented in [7–9]. Since that, the analysis is extended to DDEs with variable lags [3, 15]

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and Volterra functional differential equations (VFDEs) [1, 2, 16–20]. However, most of the existing dissipativity results of numerical methods for VFDEs (including DDEs) is independent of the size of the absorbing set of the underlying system [1, 2, 7–9, 15, 17–20]. More precisely, the radius of the absorbing set of the discrete system does not really reflect that of the underlying system. It is well known that there are results of numerical solutions which reveal the closely relationship between the two radius in the case of ODEs [14]. Let us recall the results obtained by Humphries and Stuart [11]. Consider the differential equation

$$y' = g(y), t > 0, \quad y(0) = y_0 \in \mathbb{C}^d, \tag{1.1}$$

where $g : \mathbb{C}^d \to \mathbb{C}^d$ is a locally Lipschitz continuous function such that, for some $\gamma \geq 0, \alpha < 0$,

$$\Re\langle u,\ g(u)\rangle \leq \gamma + \alpha\|u\|^2. \tag{1.2}$$

It is easy to verify that (1.1) is dissipative with absorbing set

$$B_1 = B\left(0,\ \sqrt{\frac{\gamma}{-\alpha} + \epsilon}\right) \tag{1.3}$$

for any $\epsilon > 0$ (see [14]). Humphries and Stuart [11] proved that, for any fixed step-size $h > 0$, the map generated by a DJ-irreducible, algebraically stable Runge-Kutta method is dissipative and the open ball

$$B_2 = B\left(0,\ \sqrt{\frac{\gamma}{-\alpha} + hC(\epsilon, h) + \epsilon}\right) \tag{1.4}$$

is an absorbing set for any $\epsilon > 0$, where $C(\epsilon, h)$ is a nonnegative continuous increasing function in both h and ϵ. (1.3) and (1.4) tell us that the absorbing set of the discrete system is relevant to that of the underlying system. In contrast to the case of ODEs, majority of numerical analysis for VFDEs (including DDEs) tells us nothing about the relationship between the two absorbing sets corresponding to the discrete system and the continuous system. To establish such a relationship, we [4] investigated dissipativity of the backward Euler method for DDEs (2.1a)-(2.2) and proved the backward Euler method is dissipative with absorbing set

$$\bar{B}_2 = B\left(0,\ \sqrt{\frac{\gamma}{-(\alpha+\beta)} \cdot \frac{1 - 2h\alpha}{1 - 2h(\alpha+\beta)} + \epsilon}\right) \tag{1.5}$$

for any $\epsilon > 0$. Comparing with the absorbing set of the system (2.1a)-(2.2)

$$\bar{B}_1 = B\left(0,\ \sqrt{\frac{\gamma}{-(\alpha+\beta)} + \epsilon}\right), \tag{1.6}$$

we note that the radius of the ball $\bar{B}_2$ approaches to that of the ball $\bar{B}_1$ as step-size approaches to zero.

The aim of this paper is to investigate dissipativity of one-leg θ-methods and linear θ-methods. The main results of this paper could be summarized as follows.