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Numerical Analysis of Partially Penalized Immersed Finite Element Methods for Hyperbolic Interface Problems

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Abstract. We consider an approximation of second-order hyperbolic interface problems by partially penalized immersed finite element methods. In order to penalize the discontinuity of IFE functions, we add some stabilization terms at interface edges. Some semi-discrete and fully discrete schemes are presented and analyzed. We prove that the approximate solutions have optimal convergence rate in an energy norm. Numerical results not only validate the theoretical error estimates, but also indicate that our methods have smaller point-wise error over interface elements than classical IFE methods.

AMS subject classifications: 65M15, 65M60

Key words: Hyperbolic interface problems, Cartesian mesh methods, partially penalized immersed finite element, error estimation.

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1. Introduction

Suppose that Ω is a rectangular domain or a union of several rectangular domains in $\mathbb{R}^2$. A smooth curve $\Gamma \subset \Omega$ separates Ω into two sub-domains Ω^- and Ω^+ with $\Omega = \Omega^- \cup \Omega^+ \cup \Gamma$, see Fig. 1. Such curve is usually called an interface. We shall consider the approximation of the initial-boundary-value problem for second-order hyperbolic equation defined on Ω

$$\frac{\partial^2 u}{\partial t^2} - \nabla \cdot (\beta \nabla u) = f(x,y,t), \quad (x,y) \in \Omega, \qquad t \in (0,T], \tag{1.1a}$$

$$u = 0, \qquad (x,y) \in \partial\Omega, \quad t \in (0,T], \tag{1.1b}$$

$$u(x,y,0) = u_0(x,y), \qquad u_t(x,y,0) = u_1(x,y), \quad (x,y) \in \Omega. \tag{1.1c}$$

Here, the coefficient β may be discontinuous across the interface Γ. In this article, we assume that β is the following piecewise constant function

$$\beta(x,y) = \begin{cases} \beta^-, & (x,y) \in \Omega^-, \\ \beta^+, & (x,y) \in \Omega^+ \end{cases}$$

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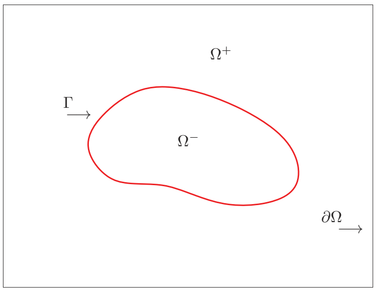

Figure 1: The simulation domain $\Omega=\Omega^-\cup\Omega^+\cup\Gamma$.

with $\min\{\beta^-,\beta^+\}>0$. For a function defined on Ω, we define its jump across the interface by

$$[[v]]_\Gamma=(v|_{\Omega^+})|_\Gamma-(v|_{\Omega^-})|_\Gamma.$$

Also, we denote by $\mathbf{n}_\Gamma$ the unit normal vector of Γ pointing from Ω^+ to Ω^-. Then, jump conditions for the exact solution u are prescribed on the interface Γ such as

$$[[u]]_\Gamma=0, \tag{1.2a}$$

$$\big[\big[\beta\nabla u\cdot\mathbf{n}_\Gamma\big]\big]_\Gamma=0. \tag{1.2b}$$

Interface problems arise in numerous applications of engineering and sciences. In the past three decades, a substantial body of research has been devoted to approximate interface problems. Among these methods, immersed finite element (IFE) methods have attracted a great deal of attention due to its advantage of allowing the use of meshes independent of interface. Especially, Cartesian meshes can be chosen if desired. Therefore, immersed finite element methods have been applied to approximate elliptic interface problems, elasticity interface problems and parabolic interface problems, etc., see [3–7, 9–17, 20, 21].

IFE functions are constructed to satisfy or weakly satisfy the jump conditions on those interface elements. Although these IFE functions are continuous within each interface element, they are usually discontinuous across interface edges. Such discontinuity may have negative impact on the global convergence. Actually, it has been reported that the errors near the interface are notable compared with other places when classical IFE methods are used and orders of convergence in both H^1 and L^2 norms can sometimes deteriorate when the mesh size becomes very small, see [15, 21]. In order to overcome this disadvantage, some researchers introduced partial penalty idea into IFE methods by adding some stabilization terms on interface edges. A partially penalized immersed finite element (PPIFE) formulation for elliptic interface problems was introduced in [15,21] and analyzed in [15]. They also provided numerical results which showed that the PPIFE methods can effectively reduce errors around interfaces. The authors in [1] introduced a partial penalty immersed finite element (PIFE) methods for anisotropic elliptic interface problems. It is deserved to be mentioned that such technique doesn't increase the total degrees of freedom and the computational cost for generating those additional stabilization terms is negligible since the number of interface edges is much smaller than the number of all cells of the mesh.