Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2018.m1622

Vol. **11**, No. 2, pp. 235-246
May 2018

Semi-Convergence Analysis of Uzawa Splitting Iteration Method for Singular Saddle Point Problems

Jingtao Li and Changfeng Ma*

College of Mathematics and Informatics, Fujian Key Laborotary of Mathematical Analysis and Applications, Fujian Normal University, Fuzhou, 350117, China

Received 22 June 2016; Accepted (in revised version) 5 August 2017

Abstract. In this paper, we propose the Uzawa splitting iteration method for solving a class of singular saddle point problems. The semi-convergence of the Uzawa splitting iteration method is carefully analyzed, which shows that the iteration sequence generated by this method converges to a solution of the singular saddle point problems under certain conditions. Moreover, the characteristics of the eigenvalues and eigenvectors of the iteration matrix of the proposed method are studied. The theoretical results are supported by the numerical experiments, which implies that Uzawa splitting iteration method is effective and feasible for solving singular saddle point problems.

AMS subject classifications: 65F10, 65F08, 65F50

Key words: Singular saddle point problems, Uzawa splitting iteration method, semi-convergence.

1. Introduction

In this paper, we consider the following large and sparse saddle point problems of the form:

$$\mathscr{A} z = \begin{pmatrix} A & B \\ -B^\top & \mathscr{O} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} p \\ -q \end{pmatrix} = f, \tag{1.1}$$

where $A \in R^{m\times m}$ is a symmetric positive definite matrix, $B \in R^{m\times n}$ $(m \geq n)$ is a rectangular matrix, $p \in R^m$ and $q \in R^n$ are two given vectors. When rank$(B) < n$, the saddle point system (1.1) is a singular saddle point system, since the coefficient matrix $\mathscr{A}$ is singular. At the moment, we call (1.1) a singular saddle point problem. Moreover, we suppose that the singular saddle point problems (1.1) are consistent, i.e., $f \in R(\mathscr{A})$. These assumptions guarantee the existence of the solution of the singular saddle point problems (1.1).

Singular saddle point problems arise in many scientific computing and engineering application areas. Such as computational fluid dynamics, image processing, mixed finite

*Corresponding author. *Email address:* `macf@fjnu.edu.cn` (C. F. Ma)

element approximation of elliptic partial differential equation, optimization, electronic networks, constrained least-squares problem. see [4–12] and the references therein. However, due to their indefiniteness and poor spectral properties, singular saddle point problems represent a significant challenge for solver developers. Since the coefficient matrix of the singular saddle point problems (1.1) is singular, its solution is not unique and can not be directly reversed.

In recent years, many effective iterative methods have been proposed to solve singular saddle point problems (1.1). Bai presented the Hermitian and skew-Hermitian splitting (HSS) iterative method for singular saddle point problems in [13], the author presented a necessary and sufficient condition for guaranteeing the semi-convergence of the HSS iterative method. Chao and chen in [14] proposed the Uzawa-SOR methods for singular saddle point problems, the authors also proved the semi-convergent of the proposed methods under certain conditions. Zhang [17] at al. studied the semi-convergence of the inexact Uzawa methods. Many other efficient iterative methods have been proposed for singular saddle point problems in the literatures, including PHSS and PAHSS iterative method [16, 18], the HSS-like methods [15], Krylov subspace methods [23], matrix splitting iterative method [24, 25], Uzawa-types methods [20–22] and Uzawa-HSS method [19].

Recently, Cao [3] at al. proposed the shift-splitting iteration method based on the following matrix splitting of the coefficient matrix $\mathscr{A}$

$$\mathscr{A}=\frac{1}{2}(\alpha I+\mathscr{A})-\frac{1}{2}(\alpha I-\mathscr{A}). \tag{1.2}$$

Inspired by this idea, in this paper, we propose the Uzawa splitting iteration method for the singular linear system (1.1). We call the new method as US iteration method for simplicity. We discussed the semi-convergence and the distribution of the eigenvalues of the iteration matrix of the US iteration method. Numerical experiments are presented to verify the theoretical results and illustrate the effectiveness of the proposed method.

The rest of paper is organized as follows. In Section 2, we propose the US iteration method to solve singular saddle point problems (1.1). The semi-convergence of the US iteration method and the properties of the eigenvalues of the iteration matrix of the proposed method are analysed in Section 3. Moreover, numerical experiments are presented to illustrate the effectiveness of US iteration method in Section 4. Finally, we draw the conclusions in Section 5.

2. The US iteration method

In this section, we will introduce the US iteration method for solving the singular saddle point problems (1.1). Let A be split as

$$A=\frac{1}{2}\big(\alpha I+A\big)-\frac{1}{2}\big(\alpha I-A\big),$$

where $\alpha>0$ is a constant and I is the identity matrix with appropriate dimension. For given $x^{(k)}\in R^m$ and $y^{(k)}\in R^n$, we apply first the iteration method below to solve the first