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A Meshless and Parallelizable Method for Differential Equations with Time-Delay

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Abstract. Numerical computation plays an important role in the study of differential equations with time-delay, because a simple and explicit analytic solution is usually un available. Time-stepping methods based on discretizing the temporal derivative with some step-size Δt are the main tools for this task. To get accurate numerical solutions, in many cases it is necessary to require $\Delta t < \tau$ and this will be a rather unwelcome restriction when τ, the quantity of time-delay, is small. In this paper, we propose a method for a class of time-delay problems, which is completely meshless. The idea lies in representing the solution by its Laplace inverse transform along a carefully designed contour in the complex plane and then approximating the contour integral by the Filon-Clenshaw-Curtis (FCC) quadrature in a few fast growing subintervals. The computations of the solution for all time points of interest are naturally parallelizable and for each time point the implementations of the FCC quadrature in all subintervals are also parallelizable. For each time point and each subinterval, the FCC quadrature can be implemented by fast Fourier transform. Numerical results are given to check the efficiency of the proposed method.

AMS subject classifications: 65M15, 65D05, 65D30

Key words: Delay differential equations, meshless/parallel computatio, contour integral, Filon-Clenshaw-Curtis quadrature.

1. Introduction

Delay differential equations (DDEs) arise from various applications, like biology [27], control of dynamic systems [5, 21], circuit engineering [32] and many others. A DDE differs from an ordinary differential equation (ODE) in that it depends not only on the

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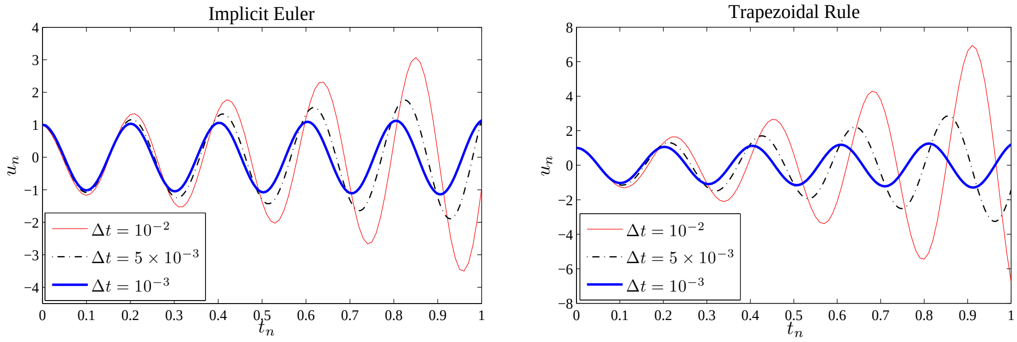

Figure 1: For DDE (1.1) with $\tau = 0.05$, the approximation of the numerical solutions generated by the implicit Euler method (left) and the trapezoidal rule (right) to the exact solution, the cosine function $u(t) = \cos(\frac{\pi}{2\tau}t)$.

solution at a present stage but also on the solution at some past stages. While the theoretical properties of DDEs have been investigated deeply and widely in the past, there is little experience with numerical methods for solving these equations, particularly with parallel computational methods. Existing numerical methods for DDEs mainly rely on discretizing the temporal derivatives with some step-size Δt by finite difference formula and then advancing the discrete solutions step by step. For this issue, we refer the interested reader to [3,4,13,14,34] and references therein. A serious limitation of the time-stepping methods is that, in many cases the step-size has to be less than the quantity of time-delay, i.e., $\Delta t < \tau$. To illustrate this, we consider the following DDE:

$$u'(t) + \frac{\pi}{2\tau}u(t-\tau) = 0, \qquad u(t) = \cos\left(\frac{\pi}{2\tau}t\right), \quad \text{for } t \le 0, \tag{1.1}$$

for which the exact solution is the cosine function $u(t) = \cos\left(\frac{\pi}{2\tau}\right)$. Let $\tau = 0.05$. Then, using the implicit Euler method and the trapezoidal rule, we get the numerical solutions for different step-sizes Δt as shown in Fig. 1. These results imply that for DDEs a small step-size is really necessary to get accurate numerical solutions when τ is small.

Parallel methods are also studied for DDEs in the past few years. Among these, we mention [11,17,33] for the study of classical *waveform relaxation* methods and [22,28–30] for the study of the Schwarz waveform relaxation methods. (The parallelism of the former is based on system decoupling, while for the latter the parallelism is based on domain decomposition.) These methods are iterative and the convergence rates of iterations usually deteriorate as the mesh parameters become small (except a coarse grid correction is used after each iteration [20,25,26]). What make matters worse is that these methods can not be directly used to compute the solution of DDEs, because they are defined at the continuous level, while in practice we still need to rely on time-stepping methods to construct discrete (Schwarz) waveform relaxation methods.

The goal of this paper is to propose a meshless and highly parallelizable method for solving the following representative model problem

$$\partial_t u(\mathbf{x},t) - \Delta u(\mathbf{x},t) + au(\mathbf{x},t-\tau) = f(\mathbf{x},t), \quad (\mathbf{x},t) \in \Omega \times (0,T), \tag{1.2}$$