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# Anisotropic Mesh Adaptation for 3D Anisotropic Diffusion Problems with Application to Fractured Reservoir Simulation

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**Abstract.** Anisotropic mesh adaptation is studied for linear finite element solution of 3D anisotropic diffusion problems. The $\mathbb{M}$-uniform mesh approach is used, where an anisotropic adaptive mesh is generated as a uniform one in the metric specified by a tensor. In addition to mesh adaptation, preservation of the maximum principle is also studied. Some new sufficient conditions for maximum principle preservation are developed, and a mesh quality measure is defined to server as a good indicator. Four different metric tensors are investigated: one is the identity matrix, one focuses on minimizing an error bound, another one on preservation of the maximum principle, while the fourth combines both. Numerical examples show that these metric tensors serve their purposes. Particularly, the fourth leads to meshes that improve the satisfaction of the maximum principle by the finite element solution while concentrating elements in regions where the error is large. Application of the anisotropic mesh adaptation to fractured reservoir simulation in petroleum engineering is also investigated, where unphysical solutions can occur and mesh adaptation can help improving the satisfaction of the maximum principle.

**AMS subject classifications**: 65M60, 65M50

**Key words**: Finite element method, anisotropic mesh adaptation, three dimensional, anisotropic diffusion, discrete maximum principle, petroleum engineering.

## 1. Introduction

We are concerned with the linear finite element solution of the three dimensional boundary value problem (BVP) of the diffusion equation

$$-\nabla\cdot(\mathbb{D}\,\nabla u)=f, \quad \text{in} \quad \Omega \tag{1.1}$$

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subject to the Dirichlet boundary condition

$$u = g, \quad \text{on} \quad \partial\Omega, \tag{1.2}$$

where $\Omega$ is a bounded polyhedral domain, $f$ and $g$ are given functions, and $\mathbb{D}$ is the diffusion matrix. We assume that $\mathbb{D} = \mathbb{D}(\boldsymbol{x})$ is a symmetric and uniformly positive definite matrix-valued function on $\Omega$. It includes both isotropic and anisotropic diffusion as special examples. In the former case, $\mathbb{D}$ takes the form $\mathbb{D} = \alpha(\boldsymbol{x})I$, where $I$ is the $3 \times 3$ identity matrix and $\alpha = \alpha(\boldsymbol{x})$ is a scalar function. In the latter case, on the other hand, $\mathbb{D}$ has not-all-equal eigenvalues at least on a portion of $\Omega$.

Anisotropic diffusion problems arise from various branches of science and engineering, including plasma physics [21–23, 51, 57], petroleum engineering [1, 2, 15, 50], and image processing [10, 11, 34, 54, 62]. When a conventional numerical method is used to solve the problems, spurious oscillations may occur in computed solutions. Numerous research has been done for two dimensional (2D) problems; among other works, we mention a few here, [8, 12, 18, 23, 32, 33, 38, 40–44, 55, 61, 63, 65, 66]. A common approach is to design a proper discretization method and/or a proper mesh so that the numerical solution satisfies the maximum principle (MP). Recently, an anisotropic non-obtuse angle condition was developed in [31,40,41,46] for the linear finite element solution of both time independent and time dependent anisotropic diffusion problems to satisfy MP.

On the other hand, much less work has been done for 3D anisotropic diffusion problems. Although MP preservation has been studied in general dimensions e.g. in [8, 12, 23, 33, 37, 38, 40, 41], most of them either consider isotropic diffusion or present numerical examples only in 1D and 2D. For example, only isotropic diffusion is considered in [12,37]. It is shown in [39] that the 3D Delaunay triangulation does not generally produce a mesh with which the numerical solution satisfies MP. Mesh conditions are studied in [8] for a reaction-isotropic-diffusion problem for general dimensions and numerical examples in 1D and 2D are presented. The difficulty of MP satisfaction for 3D problems is remarked in both [8] and [39].

The objective of this paper is to study the linear finite element solution of 3D anisotropic diffusion problems. The focus will be on MP preservation and mesh adaptation. Four different metric tensors used in anisotropic mesh generation will be considered. This study is a 3D extension of the work [40]. Moreover, new sufficient conditions will be developed for the linear finite element approximation to satisfies MP, and a mesh quality measure will be defined to provide a useful indication for MP satisfaction. The mesh quality measure is developed along the approach of [28]. But the interested reader is also referred to [35,36] and references therein for different mesh optimization methods and quality metrics. Furthermore, the application to fractured reservoir simulation in petroleum engineering will also be investigated, where unphysical solutions can occur and mesh adaptation can help improving MP satisfaction.

An outline of the paper is given as follows. The linear finite element formulation for BVP (1.1) and (1.2) is given in Section 2. MP preservation and some sufficient conditions will be discussed. Section 3 contains the discussion of anisotropic mesh adaptation and