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Numerical Effects of the Gaussian Recursive Filters in Solving Linear Systems in the 3Dvar Case Study

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Abstract. In many applications, the Gaussian convolution is approximately computed by means of recursive filters, with a significant improvement of computational efficiency. We are interested in theoretical and numerical issues related to such an use of recursive filters in a three-dimensional variational data assimilation (3Dvar) scheme as it appears in the software OceanVar. In that context, the main numerical problem consists in solving large linear systems with high efficiency, so that an iterative solver, namely the conjugate gradient method, is equipped with a recursive filter in order to compute matrix-vector multiplications that in fact are Gaussian convolutions. Here we present an error analysis that gives effective bounds for the perturbation on the solution of such linear systems, when is computed by means of recursive filters. We first prove that such a solution can be seen as the exact solution of a perturbed linear system. Then we study the related perturbation on the solution and we demonstrate that it can be bounded in terms of the difference between the two linear operators associated to the Gaussian convolution and the recursive filter, respectively. Moreover, we show through numerical experiments that the error on the solution, which exhibits a kind of edge effect, i.e. most of the error is localized in the first and last few entries of the computed solution, is due to the structure of the difference of the two linear operators.

AMS subject classifications: 65G20, 65Y04, 62H99

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1. Introduction

In recent years, Gaussian filters have achieved a central role in image filtering and techniques for accurate measurement [1]. The implementation of a Gaussian filter in one or more dimensions has typically been done as a convolution with a Gaussian kernel,

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which leads to a very high computational cost in practical applications. Several efforts to reduce Gaussian convolution computational complexity are discussed in [2, 3]. Great advantage may be gained by employing a *recursive filter (RF)*, carefully constructed to mimic the Gaussian convolution operator, initially developed in the field of time series analysis, RFs are now extensively used as computational tools in numerical weather analysis and forecasting [4–6], digital image processing [7, 8], biomedical signal processing [9–11]. In this paper, we focus on the numerical effects of the RFs technique when applied to a three-dimensional variational data assimilation (3Dvar) scheme, as it appears in the oceanographic tool OceanVar [12]. In [13], the benefits of using RFs in OceanVar, with emphasis on the results of an overall oceanographic forecast, are discussed. The most costly numerical problem in OceanVar consists in solving a large linear system. The solution is computed by means of the Conjugate Gradient (CG) method whose main operation is a matrix-vector multiplication, which turns out to be a Gaussian convolution. In OceanVar the latter operation is approximated by RFs. Preliminary results about such an application of RFs are given in [14], whose main result was the establishing of a bound for the error at each step of the CG, due to the use of the RFs. Here, a more general analysis is carried out with the aim to study how such RFs, when are used instead of the Gaussian convolution, affect the computed solution of the linear system. This study relies on a perturbation analysis approach and shows that the computed solution is in fact the exact solution of a perturbed linear system. Then it is proved that the related perturbation on the solution can be bounded in terms of the difference between the two linear operators associated to the Gaussian convolution and the recursive filter, respectively. Finally, we also provide some insight about the pattern of the error on the solution computed by means of the RFs, by highlighting the presence of the so-called edge effects. The latter phenomenon is known to 3Dvar practitioners, and we give evidence that it is a direct consequence of the peculiar structure of the difference of the two linear operators.

The rest of the paper is organized as follows. In Section 2, we recall the three-dimensional variational data assimilation problem, as it appears in OceanVar, and describe the underlying linear system that is solved by means of the CG method. In Section 3, we introduce the K-iterated n-order Gaussian RFs and derive some properties that are used in the sequel. In Section 4, we provide a comprehensive perturbation error analysis for the solution of the linear system; moreover, with regard to the effects of the RFs on the CG method, we give an effective and computable bound estimate that extends a result we derived elsewhere. In Section 5, we report some experiments to confirm our theoretical results and to show and discuss the edge effects on the computed solution. Finally, Section 6 contains some concluding remark.

2. Preliminaries on the 3Dvar problem

The aim of a variational data assimilation problem is to find the best estimate x, given a previous estimate x^b and a measured value y. Such a problem can be stated as the