Numer. Math. Theor. Meth. Appl. Vol. **10**, No. 2, pp. 465-488
doi: 10.4208/nmtma.2017.s12 May 2017

A Stochastic Galerkin Method for the Boltzmann Equation with Multi-Dimensional Random Inputs Using Sparse Wavelet Bases

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Received 19 October 2016; Accepted (in revised version) 15 January 2017

Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. We propose a stochastic Galerkin method using sparse wavelet bases for the Boltzmann equation with multi-dimensional random inputs. The method uses locally supported piecewise polynomials as an orthonormal basis of the random space. By a sparse approach, only a moderate number of basis functions is required to achieve good accuracy in multi-dimensional random spaces. We discover a sparse structure of a set of basis-related coefficients, which allows us to accelerate the computation of the collision operator. Regularity of the solution of the Boltzmann equation in the random space and an accuracy result of the stochastic Galerkin method are proved in multi-dimensional cases. The efficiency of the method is illustrated by numerical examples with uncertainties from the initial data, boundary data and collision kernel.

AMS subject classifications: 35Q20, 65L60

Key words: Uncertainty quantification, Boltzmann equation, stochastic Galerkin methods, sparse wavelets.

1. Introduction

The Boltzmann equation plays an essential role in kinetic theory [9]. It describes the time evolution of the density distribution of dilute gases, where fluid dynamics equations, such as the Euler equations and the Navier-Stokes equations, fail to provide

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reliable information. It is an indispensable tool in fields concerning non-equilibrium statistical mechanics, such as rarefied gas dynamics and astronautical engineering.

For most applications of the Boltzmann equation, the initial and boundary data are given by physical measurements, which inevitably bring measurement errors. Furthermore, due to the difficulty of deriving the collision kernels from first principles, empirical collision kernels are often used. Such kernels contain adjustable parameters which are determined by matching with experimental data [5]. This procedure involves uncertainty on the parameters in the collision kernel. To understand the impact of these random inputs on the solution of the Boltzmann equation, it is imperative to incorporate the uncertainties into the equation, and design numerical methods to solve the resulting system [30]. A proper quantification of uncertainty will provide reliable predictions and a guidance for improving the models. Since the uncertainties of the Boltzmann equations come from many independent sources, it is necessary to use a multi-dimensional random space to incorporate all the uncertainties. Moreover, a Karhunen-Loeve expansion of a random field will result in a multi-dimensional random space.

Various numerical methods have been developed to solve the problem of uncertainty quantification (UQ) [12, 19, 30, 31]. Monte-Carlo methods [23] use statistical sampling in the random space, which give halfth order convergence in any dimension. Stochastic collocation methods [2, 4, 22] take sampling points on a well-designed grid, usually according to a quadrature rule, or take sampling points by least-square or compressed sensing approaches, and the statistical moments are computed by numerical quadratures or reconstructed generalized polynomial chaos expansions. Stochastic Galerkin methods [3, 4] use an orthonormal basis expansion in the random space. By a truncation of the expansion and Galerkin projection, one is led to a deterministic system of expansion coefficients. Both methods can achieve spectral accuracy in one-dimensional random space if the quadrature rule (orthonormal basis) is well chosen.

Hu and Jin [16] gave a first numerical method to solve the Boltzmann equation with uncertainty by a generalized polynomial chaos based stochastic Galerkin method. By a singular value decomposition on a set of basis related coefficients, together with the fast spectral method for the Boltzmann collision operator proposed by [21], the computational cost of the collision operator is decreased dramatically. However, their work focuses on low dimensional random spaces, and a direct extension of their method to multi-dimensional random spaces will suffer from the curse of dimensionality, which means K, the total number of basis functions, will grow like $K=\binom{K_1+d}{K_1}$, where K_1 is the number of basis in one dimension, and d is the dimension of the random space. This cost is not affordable if both K_1 and d are large. Monte-Carlo methods are feasible, but a halfth order convergence rate can be unsatisfactory in many applications. Therefore it is desirable to have an efficient and accurate method to solve the Boltzmann equation with multi-dimensional random inputs.

In this work, we adopt a sparse approach [8, 11] for the stochastic Galerkin method to circumvent the curse of dimensionality. The idea of sparse approaches traces back to Smolyak [28]. In recent years, sparse approaches have become a major way to break

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