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Runge-Kutta Discontinuous Local Evolution Galerkin Methods for the Shallow Water Equations on the Cubed-Sphere Grid

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Dedicated to Professor Zhenhuan Teng on the occasion of his 80th birthday

Abstract. The paper develops high order accurate Runge-Kutta discontinuous local evolution Galerkin (RKDLEG) methods on the cubed-sphere grid for the shallow water equations (SWEs). Instead of using the dimensional splitting method or solving one-dimensional Riemann problem in the direction normal to the cell interface, the RKDLEG methods are built on genuinely multi-dimensional approximate local evolution operator of the locally linearized SWEs on a sphere by considering all bicharacteristic directions. Several numerical experiments are conducted to demonstrate the accuracy and performance of our RKDLEG methods, in comparison to the Runge-Kutta discontinuous Galerkin method with Godunov's flux etc.

AMS subject classifications: 65M25, 76M10

Key words: RKDLEG method, evolution operator, genuinely multi-dimensional method, shallow water equations, cubed-sphere grid.

1. Introduction

The shallow water equations (SWEs) describe the motion of a thin layer of fluid held down by gravity. The SWEs on the sphere exhibit the major difficulties associated with the horizontal dynamical aspects of atmospheric modeling on the spherical earth and thus are important in studying the dynamics of large-scale atmospheric flows and developing numerical methods of more complex atmospheric models. In comparison with the planar case, the difficulties in solving the SWEs on the sphere mainly come from the spherical geometry, the choice of coordinates, nonlinearity, and the large scale difference between the horizontal and vertical motions of the fluids. High-order accurate

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numerical methods are becoming increasingly popular in atmospheric modeling, but the numerical methods should be competent for long time simulation. In order to evaluate numerical methods for the solutions of SWEs in spherical geometry, Williamson et al. proposed a suite of seven test cases and offered reference solutions to those tests obtained by using a pseudo-spectral method [50].

Representation of the spherical geometry plays an important role in solving the SWEs on the sphere. The latitude-longitude (LAT/LON) coordinates or grids are naturally and popularly chosen in the early stage [2, 24, 31], but the singularity at the poles leads to big numerical difficulty. Overcoming such pole singularity needs special numerical technique and boundary conditions [39]. To avoid the pole singularity in the LAT/LON coordinates, other choices are the icosahedral hexagonal or triangular grids [19, 21, 36, 48], Yin-Yang grid [18, 22, 23], and cubed-sphere grid [7, 34, 37–39, 49]. Comparisons of those frequently-used grids are given in [6, 42]. An icosahedral-hexagonal grid on the sphere is created by dividing the faces of an icosahedron and projecting the vertices onto the sphere, thus it is non-quadrilateral and unstructured. The Ying-Yang grid is overset in spherical geometry and composes of two identical component grids combined in a complementary way to cover a spherical surface with partial overlap on their boundaries so that the interpolation should be used between two component grids. The cubed-sphere grid is quasi-uniform and easily generated by dividing the sphere into six identical regions with the aid of projection of the sides of a circumscribed cube onto a spherical surface and choosing the coordinate lines on each region to be arcs of great circles. The mainly existing numerical methods for the SWEs on the sphere are as follows: finite-difference [2, 39, 47, 48], finite-volume [21, 24, 52], multi-moment finite volume [6, 7, 22, 23], spectral transform [16], spectral element [12, 44, 46], and discontinuous Galerkin (DG) methods [11, 13, 19, 34, 35] etc. Most of them are built on the one-dimensional exact or approximate Riemann solver.

The aim of the paper is to develop Runge-Kutta discontinuous local evolution Galerkin (RKDLEG) methods for the SWEs on the cubed sphere. They are the (genuinely) multi-dimensional and combining the Runge-Kutta discontinuous Galerkin (RKDG) methods on the cubed-sphere with the local evolution Galerkin (LEG) method, which is a modification and simplification of the original finite volume evolution Galerkin (EG) method for nonlinear multi-dimensional hyperbolic system [30, 43]. The EG method generalizes the Godunov method with an evolution operator coupling the flux formulation of each direction type for the multi-dimensional hyperbolic system. The basic idea of the EG method was introduced in [32], and then it was developed for the linear hyperbolic system in [27] and nonlinear hyperbolic systems in [26, 30]. The EG method is constructed by using the theory of bicharacteristics in order to take all infinitely many directions of wave propagation into account and give the exact and approximate evolution operators of the linearized hyperbolic system, in other words, integrating the linearized hyperbolic system along its bicharacteristics to obtain an equivalent integral system, then making a suitable approximations of the integral system. Similar bicharacteristic-type methods for hyperbolic system can be found in the early literature such as [5, 14, 17]. The EG method may be considered as a genuinely