Numer. Math. Theor. Meth. Appl.
doi: 10.4208/nmtma.2017.m1611

Vol. **10**, No. 1, pp. 98-115
February 2017

Fast Linearized Augmented Lagrangian Method for Euler's Elastica Model

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Received 24 February 2016; Accepted 8 July 2016

Abstract. Recently, many variational models involving high order derivatives have been widely used in image processing, because they can reduce staircase effects during noise elimination. However, it is very challenging to construct efficient algorithms to obtain the minimizers of original high order functionals. In this paper, we propose a new linearized augmented Lagrangian method for Euler's elastica image denoising model. We detail the procedures of finding the saddle-points of the augmented Lagrangian functional. Instead of solving associated linear systems by FFT or linear iterative methods (e.g., the Gauss-Seidel method), we adopt a linearized strategy to get an iteration sequence so as to reduce computational cost. In addition, we give some simple complexity analysis for the proposed method. Experimental results with comparison to the previous method are supplied to demonstrate the efficiency of the proposed method, and indicate that such a linearized augmented Lagrangian method is more suitable to deal with large-sized images.

AMS subject classifications: 65M55; 68U10; 94A08

Key words: Image denoising, Euler's elastica model, linearized augmented Lagrangian method, shrink operator, closed form solution.

1. Introduction

Image denoising aims to recover a noise-free image u from a noise-polluted image f. In general, it can be modeled by $f = u + \eta$, where η is the unknown noise. It has been a challenging topic and has been deeply investigated during the last two decades.

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Excellent results have been obtained by using the total variation (TV) model, which involves solving a second-order partial differential equation (PDE). This model was proposed by Rudin, Osher and Fatemi [21] (called the ROF model). Recently, the gap between a continuous ROF model and its discretized version was studied in [14].

The traditional way to solve the ROF model is to solve the corresponding Euler-Lagrange equation with some time marching methods [25]. Due to the stability constraint on time step size, this kind of methods usually converges slowly. To overcome the non-differentiability of the ROF model and reduce computational cost, several fast algorithms have been studied. For example, the idea of duality was first proposed by Chan *et al.* [5], later by Carter [2] and Chambolle [3, 4], and then by Zhu [29], Zhu and Chan [30], Zhu *et al.* [31]. In [11], Goldstein and Osher developed a split Bregman method where the Gauss-Seidel iteration was used to solve some linear systems. If the periodic boundary condition is used, such linear systems generated by the split Bregman method can be solved by fast Fourier transform (FFT) [27]. In [24], the authors introduced an augmented Lagrangian method for solving the ROF model. More specifically, it was shown in [26] that the CGM method, Chambolle's dual algorithm, the split Bregman method and the augmented Lagrangian method were equivalent.

Although the CGM method, the split Bregman method and the augmented Lagrangian method can deal with the ROF model efficiently, associated linear systems still need to be solved by FFT or the Gauss-Seidel method. In order to avoid solving a linear equation system and simultaneously obtain a closed form solution, Jia *et al.* proposed an efficient algorithm by applying matrix-vector multiplication [16]. Based on the same idea, that is, avoiding solving PDEs by FFT or linear iterative methods, Duan and Huang adopted a fixed-point strategy and proposed a fixed-point augmented Lagrangian method for TV minimization problems [9].

By using the ROF model, the edges and the small scale characteristics can be preserved during noise elimination. So this model has been extensively used for a variety of image restoration problems (see [6, 20]). However, the ROF model often causes staircase effects in the results. To overcome this difficulty, some high order models [8, 17–19, 22, 28, 32] have been introduced in image restoration recently. Also, Hahn *et al.* did some research on reducing staircase effects [1, 12, 13]. In [22], the following minimization problem based on a fourth-order PDE (called the Euler's elastica model) has been proposed:

$$\min_u \int_\Omega \left(a + b \left(\nabla \cdot \frac{\nabla u}{\mid \nabla u \mid} \right)^2 \right) \mid \nabla u \mid + \frac{\lambda}{2} \int_\Omega (u - f)^2, \tag{1.1}$$

where $\lambda > 0$ is a weighting parameter which controls the amount of denoising, and the term $\nabla \cdot \frac{\nabla u}{|\nabla u|}$ is the curvature of level curve $u(x,y) = c$. The use of fourth-order derivatives damps out high frequency components of images faster than the second-order PDE-based methods, so (1.1) can reduce the staircase effects and produce better approximation to the nature image. Indeed, it is also able to preserve object edges while erasing noise. The Euler's elastica model is one of the most important high order