Numer. Math. Theor. Meth. Appl. Vol. **13**, No. 1, pp. 253-280
doi: 10.4208/nmtma.OA-2019-0038 February 2020

# The Conservation and Convergence of Two Finite Difference Schemes for KdV Equations with Initial and Boundary Value Conditions

Jinye Shen, Xuping Wang and Zhi-zhong Sun*

*School of Mathematics, Southeast University, Nanjing 210096, Jiangsu, China*

Received 8 March 2019; Accepted (in revised version) 28 August 2019

**Abstract.** Korteweg-de Vries equation is a nonlinear evolutionary partial differential equation that is of third order in space. For the approximation to this equation with the initial and boundary value conditions using the finite difference method, the difficulty is how to construct matched finite difference schemes at all the inner grid points. In this paper, two finite difference schemes are constructed for the problem. The accuracy is second-order in time and first-order in space. The first scheme is a two-level nonlinear implicit finite difference scheme and the second one is a three-level linearized finite difference scheme. The Browder fixed point theorem is used to prove the existence of the nonlinear implicit finite difference scheme. The conservation, boundedness, stability, convergence of these schemes are discussed and analyzed by the energy method together with other techniques. The two-level nonlinear finite difference scheme is proved to be unconditionally convergent and the three-level linearized one is proved to be conditionally convergent. Some numerical examples illustrate the efficiency of the proposed finite difference schemes.

**AMS subject classifications**: 65M06, 65M12, 65M15

**Key words**: Nonlinear Korteweg-de Veries equation, difference scheme, existence, conservation, boundedness, convergence.

## 1. Introduction

In recent years, with most complex phenomenon appearing, nonlinear evolutionary equations [7, 23, 24] have become more and more important tool to describe them. Especially, Korteweg-de Vries (KdV) type equation has been widely applied in physics, mathematics, biophysics, which originated from modeling the shallow water surface height of solitary dispersive waves. In 1877, Boussinesq firstly discovered the KdV equation. After about twenty years, Korteweg and his PhD student Gustav de Vries mathematically reintroduced the KdV equation [5, 14].

*Corresponding author. *Email addresses:* goldleaf0811@sina.com (J. Y. Shen), seuMathWxp@139.com (X. P. Wang), zzsun@seu.edu.cn (Z. Z. Sun)

 

The simplest form of KdV equation is as follows:

$$u_t + \epsilon u u_x + \mu u_{xxx} = 0,$$

where $\epsilon$, $\mu$ are given constants.

There are many works on KdV equation. Kamruzzaman used the Kudryashov method to find the exact travelling wave solutions transmutable to the solitary wave solutions of the ubiquitous unsteady Korteweg-de Vries equation and applied the $\exp(-\phi(\xi))$-expansion method to construct the exact travelling wave solutions for nonlinear evolution equations [12, 13]. About the numerical approximation of the one-dimensional simplified KdV equation, many researchers have obtained abundant results by finite difference method, spectral method and finite volume method. For example, Vliegenthart discussed some explicit finite difference schemes for solving the initial-value problem of KdV equation and presented dissipative difference schemes which had the effect of eliminating high wave number components. However, these schemes were conditionally stable [22]. The authors of [15] proposed a Legendre pseudo-spectral method for the KdV equation with nonperiodic boundary condition and analyzed the convergence for linear-KdV equation. Pazoto got a one order fully-implicit numerical scheme based on this asymptotic behavior of the solution of the generalized Korteweg-de Vries equation (GKdV with $p = 4$) and got convergece in $L_4$ norm [16]. Dougalis constructed a fully discrete Galerkin method with high order of accuracy for the numerical solution of the periodic initial-value problem for KdV equation. But they needed certain mild restriction on the space mesh length and the time step [9]. Alisamii et al. introduced a hybridized discontinuous Galerkin method to deal with nonlinear KdV type equations. As for the time stepping, they used the backward difference formula. However, there was no analysis of convergence in this research [2]. Yan proposed three conservative finite volume element schemes based on the discrete variational derivative method [26], but there was no analysis of convergence. Winebery developed an implicit-stepping scheme for KdV equation in temporal direction and spectral methods in space [25]. However, there was a restriction on the size of the time step when they applied predictor-corrector method to retain the full accuracy of the scheme. In addition, Bosco presented a finite difference method for the integration of the KdV equation with periodic boundary conditions on irregular grid. The method is shown to be superconvergent, which only took place on grids with an odd number of points per period [10]. Zhu constructed a difference scheme with a higher-order discrete invariant for the periodic KdV equation [27]. Djidjeli et al. proposed two numerical methods for the solution of the third- and fifth-order KdV equations. The first method was derived using central differences to replace the space derivative with predictor-corrector time-stepping and the second method by linearizing the implicit corrector scheme in which the solution was then found by solving a linear algebraic system at each time step. They proved the stability of these schemes [8]. And Qu and Wang had presented an alternating segment explicit-implicit difference scheme and proved the stability of this scheme by the analysis of linearization procedure [17]. Nonetheless, there were lack of the convergence analysis.