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Analysis of a Special Q_1-Finite Volume Element Scheme for Anisotropic Diffusion Problems

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Abstract. In this paper, we analyze a special Q_1-finite volume element scheme which is obtained by using the midpoint rule to approximate the line integrals in the standard Q_1-finite volume element method. A necessary and sufficient condition for the positive definiteness of the element stiffness matrix is obtained. Based on this result, a sufficient condition for the coercivity of the scheme is proposed. This sufficient condition has an explicit form involving the information of the diffusion tensor and the mesh. In particular, this condition can reduce to a pure geometric one that covers some special meshes, including the parallelogram meshes, the $h^{1+\gamma}$-parallelogram meshes and some trapezoidal meshes. Moreover, the H^1 error estimate is proved rigorously without the $h^{1+\gamma}$-parallelogram assumption required by existing works. Numerical results are also presented to validate the theoretical analysis.

AMS subject classifications: 65M10, 78A48

Key words: Q_1-finite volume element scheme, midpoint rule, coercivity, H^1 error estimates.

1. Introduction

In the past several decades, the finite volume method has been one of the most common discretization methods for solving partial differential equations. One main attractive property of the finite volume method is its local conservation for physical quantities such as mass, momentum and energy. What's more, similar to the finite difference method, the finite volume method requires less computational expenses, while it can be used to solve problems with complex geometries, just like the finite element

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method. In this paper, we consider a special finite volume method, i.e., the finite volume element method (FVEM) [2, 24], which is also called as generalized difference method [3, 4, 13–15, 27], box scheme [1, 9, 22], or covolume method [6, 7, 20].

FVEM can be viewed as a certain Petrov-Galerkin finite element method. Compared with its wide applications, the mathematical theory of the FVEM is not a fine match, at least, not as satisfactory as that for the finite element method. The main reason is that the trial function space and the test function space in FVEM are different, which results in asymmetric bilinear forms. Among the theoretical results for FVEM, the coercivity is the most fundamental one. In 1987, Bank and Rose [1] first discovered that the linear FVEM for triangular meshes is closely related to the linear finite element method (FEM). From then on, treating linear FVEM as a certain perturbation of the corresponding FEM has become an efficient and popular tool in the analysis of linear FVEM [16]. For example, an unconditional coercivity result with its simple proof can be found in [25]. By comparison, it is not easy to do the relevant study for the Q_1-FVEM on quadrilateral meshes, since the relation to Q_1-FEM is not clear so far. Q_1-FVEM was first suggested in [27]. Since then, numerous work has been devoted to the analysis of this method, including H^1, L^2 and L^∞ error estimates [10, 11, 13, 15, 17–19, 26] and superconvergence [18, 21, 23]. To our knowledge, the proof of the coercivity for the Q_1-FVEM usually requires certain geometric assumptions. For example, the following assumptions are frequently seen in the literature.

(1) h^2-parallelogram assumption: $m_K \leq Ch^2$ and h is small enough;

(2) $h^{1+\gamma}$-parallelogram assumption: $m_K \leq Ch^{1+\gamma}(\gamma > 0)$ and h is small enough;

(3) $m_K \leq Ch$ and C is sufficiently small.

Here, C is a constant independent of h and m_K denotes the distance between the two diagonal midpoints of the quadrilateral cell K. The authors in [15] obtained the coercivity and H^1 error estimate of the Q_1-FVEM for Poisson equations under assumption (1). Under assumption (3), the coercivity of the Q_1-FVEM for anisotropic diffusion problems with full diffusion tensors was studied in [22] where the existence of such a C is proved. However, for a specific mesh and a specific diffusion tensor, it is difficult to judge whether assumption (3) is satisfied or not. Under assumption (2), [26] proposed a unified proof for the inf-sup condition of any order finite volume element scheme on quadrilateral meshes with scalar diffusion coefficients. Recently, the coercivity of a modified Q_1-FVEM scheme was studied in [11] for anisotropic diffusion problems on general quadrilateral meshes. This modified scheme is obtained by using the trapezoidal rule to approximate the line integrals in the bilinear finite volume element method.

In this paper, by using the midpoint rule to approximate the line integrals in the traditional Q_1-FVEM, we obtain a special finite volume scheme that is called as Q_1-FVEM-MR for short. In fact, the integrands in the line integrals of the Q_1-FVEM are generally rational functions, so that a certain quadrature method is required to evaluate these line integrals. Different quadrature rules will result in different Q_1-FVEM