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Error Estimates and Superconvergence of a High-Accuracy Difference Scheme for a Parabolic Inverse Problem with Unknown Boundary Conditions

Liping Zhou[1,2], Shi Shu[1,3] and Haiyuan Yu[1,*]

[1] *School of Mathematics and Computational Science, Xiangtan University, Xiangtan 411105, Hunan, China*
[2] *Institute of Computational Mathematics, College of Science, Hunan University of Science and Engineering, Yongzhou 425199, Hunan, China*
[3] *Hunan Key Laboratory for Computation and Simulation in Science and Engineering, Xiangtan University, Xiangtan 411105, Hunan, China*

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Abstract. In this work, we firstly construct an implicit Euler difference scheme for a one-dimensional parabolic inverse problem with a unknown time-dependent function in the boundary conditions. Then we initially prove that this scheme can reach the asymptotic optimal error estimate in the maximum norm. Next, we present some approximation formulas for the solution derivative and the unknown boundary function and prove that they have superconvergence properties. In the end, numerical experiment demonstrates the theoretical results.

AMS subject classifications: 65M06, 65M12, 65T50
Key words: Parabolic inverse problem, unknown boundary condition, finite difference method, discrete Fourier transform, asymptotic optimal order, superconvergence.

1. Introduction

The research history of parabolic inverse problems with unknown boundary conditions can be traced back to the 60s of last century [1–3,8,9]. Since these problems arise in physics and applied mathematics, in areas such as heat conduction, thermoelasticity, chemical diffusion, control theory and so on, their analytic theory and numerical methods have aroused the concern of many scholars in the last decades.

In the context of the heat conduction problem, Rundell and his collaborators [16, 17] consider the physical models that the heat flux across the boundaries at the rod is an unknown function of the temperature. In [16], the existence and uniqueness of

*Corresponding author. *Email addresses:* `lpzhuse@163.com` (L. P. Zhou), `shushi@xtu.edu.cn` (S. Shu), `HaiyuanYuXtu@163.com`, `spring@xtu.edu.cn` (H. Y. Yu)

the solution are presented and an iterative method is justified. In [17], the uniqueness and the continuous dependence of the classical solution are presented. Recently, considerable attention has been paid to the recovery of the space-wise or time-dependent ambient temperature for parabolic initial-boundary problems. Ebrahimian et al. [4] use the time discretization to change the problem to an ordinary differential equation which can be easily solved. The boundary element method(BEM) which only requires a boundary mesh to discretize these problems is found in [6, 10–12]. Many researchers have considered inverse parabolic problems in which an unknown coefficient appears on the boundary. T. Suzuki [19–21] presents the conditions of uniqueness for the solution with the method of Gel'fand-Levitan. Two popular numerical methods that one is the boundary element method [5, 13–15] and the other is the Sinc method [23], are developed. For parabolic inverse source problems with moving boundary condition, M. Slodička [18] shows the well-posedness of the solution with variation method and gives a time semi-discretization scheme based on the method of line; In [7], the numerical solution based on the method of fundamental solutions and discrepancy principle for source place choosing is obtained. However, at present, not much work has been done in design, calculation and theoretical analysis for difference schemes which can reach the asymptotic optimal error estimate or the optimal error estimate for parabolic inverse problem with unknown boundary condition. Besides, in many application problems, not only the approximation of the exact solution need to be considered, but also the approximation of the solution derivatives.

The challenge to solve these problems is that traditional error analysis methods of difference schemes are difficult to obtain high-accuracy and convergence results for parabolic inverse problems. So it is bound to develop some new methods and techniques.

In this work, we first build an implicit Euler difference scheme of a one-dimensional parabolic inverse problem with an unknown time-dependent function in boundary conditions (see [23]). Then, we introduce some new methods and techniques on basis of the discrete Fourier transform (DFT). Under a general condition $\tau \geq Ch^2$ (C is a positive constant independent of mesh size), we prove that the errors of the scheme at boundary points and interior points are $\mathcal{O}(\tau\,|\ln h|)$ and $\mathcal{O}(\tau \ln^2 h)$, respectively. Moreover, we present formulas to approximate the partial derivative of the exact solution and the unknown boundary function and prove that these formulas have the super approximations of $\mathcal{O}(\tau \ln^2 h)$ at interior points for u_x and the unknown boundary function. It is worthwhile noting that the methods and techniques presented in this work can be extended to other parabolic inverse problems with nonlocal conditions for error analysis.

This work is organized as follows. In Section 2, we depict a parabolic inverse problem with unknown boundary condition and present an implicit Euler scheme. The error equations of the scheme are analyzed with the DFT and the corresponding error estimate is presented in Section 3. Section 4 is devoted to presenting the approximation formulas for the spatial partial derivatives of the exact solution and the unknown boundary function. Besides, the corresponding superconvergence are discussed. In