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Convergence Analysis of the Inexact Uzawa Algorithm for Nonlinear Saddle Point Problems

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Abstract. In this paper, we consider the inexact Uzawa algorithm for solving the nonlinear saddle point problem. Based on the energy norm, some sufficient conditions are given for the convergence of the algorithm, and the error analysis is also presented. Moreover, some numerical results are reported to illustrate the efficiency of the considered algorithm.

AMS subject classifications: 65H10

Key words: Nonlinear saddle point problem, inexact Uzawa algorithm, preconditioned, convergence analysis, numerical experiments.

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1. Introduction

We consider the following nonlinear saddle point problem

$$\begin{cases} F(x) + B^T y = f, \\ Bx - Cy = g, \end{cases} \tag{1.1}$$

where $B \in \mathbb{R}^{m\times n}$ is a full row rank matrix ($m \le n$), B^T is the transpose of the matrix B, $C \in \mathbb{R}^{m\times m}$ is a symmetric positive semi-definite matrix, f is a vector in $\mathbb{R}^n$, g is a vector in $\mathbb{R}^m$ and $F : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear vector-valued function, not necessarily differentiable.

The nonlinear saddle point problem of the form (1.1) arises frequently in electromagnetic Maxwell equations [1, 2], partial differential equations [3, Chapter IV, Pages 278–351], nonlinear optimization [4, Section 10.2, Pages 86–88], quadratic stochastic programming [6, 7], for example, of the form (see for [5–7])

$$\begin{cases} \text{minimize } \ \dfrac{1}{2}u^T Mu + d^T u + g(v), \\ \text{s.t.} \quad Pu + Sv = h, \end{cases} \tag{1.2}$$

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where $d \in \mathbb{R}^s, u \in \mathbb{R}^s$, $M \in \mathbb{R}^{s\times s}$ is a symmetric positive definite matrix, $g : \mathbb{R}^t \to \mathbb{R}$ is a continuous differentiable convex function but not necessarily twice differentiable, P is an $m\times s$ matrix, S is an $m\times t$ matrix and h is a vector satisfying $h \in \mathbb{R}^m$. Specially, when $F(x) = Ax$ with $A \in \mathbb{R}^{n\times n}$ being a symmetric positive definite matrix, the problem (1.1) reduces to the well-known linear stablized saddle point problem [47, 48, 50]

$$\begin{cases} Ax + B^T y = f, \\ Bx - Cy = g. \end{cases} \tag{1.3}$$

The stablized linear saddle point problem arises in computational science and engineering applications, including computational fluid dynamics [8–10], constrained optimization, parameter identification [11], mixed finite element approximation of elliptic partial differential equation [12, 13], and others = [34–46]. In [51], the authors proposed Uzawa splitting iteration method for the saddle point problem (1.3) with $C = O$ and B being full row rank, and eatablished the semi-convergence of the iteration method. The Uzawa splitting iteration methods are based on the splitting of saddle point matrix, which need to choose appropriate parameters and preconditioners. During the past decade, there has been a growing interest in preconditioned iterative method for solving (1.3). Particularly, the inexact Uzawa method is the most popular iterative method for solving (1.3) (see [19–26, 31, 32, 49] and the references therein), as the inexact Uzawa-type algorithms have minimal memory requirement and are easy to implement. The inexact Uzawa methods replace the exact inverse in the Uzawa algorithm by an “incomplete” or “approximate” evaluation of A^{-1}. However, the existing methods for solving the nonlinear saddle point problem (1.1) are relatively fewer than the linear saddle point problem (1.3).

The Uzawa algorithm for solving the nonlinear saddle point problem (1.1) generates sequences $\{x_k\}$ and $\{y_k\}$ as follows (see [14, Section 9.4, Pages 360–369])

$$\begin{cases} F(x_{k+1}) = f - B^T y_k, \\ y_{k+1} = y_k + \alpha(B^T x_{k+1} - Cy_k - g), \end{cases} \tag{1.4}$$

where α is a positive stepsize. The Uzawa method (1.4) is globally convergent. However, the convergence rate is not expected to be faster than linear. Furthermore, each step of the Uzawa algorithm requires the solution of the nonlinear system, which is not easy to solve exactly.

Chen [15] gave an inexact Uzawa method for solving the nonlinear saddle problem (1.1) and obtained the global and superlinear convergence results in the standard l^2-norm. Her preconditioned inexact Uzawa algorithm is defined as follows

$$\begin{cases} F(x_{k+1}) = f - B^T y_k + \delta_k, \\ y_{k+1} = y_k + \alpha_k Q_k^{-1}(B^T x_{k+1} - Cy_k - g), \end{cases} \tag{1.5}$$

where α_k is a positive stepsize, $Q_k \in \mathbb{R}^{m\times m}$ is a symmetric positive definite matrix, and the vector δ_k is the residual of the approximate solution x_{k+1} to the system $F(x) = f - B^T y_k$. Besides, Chen [15] proposed a Newton-Uzawa hybrid method by combining the