NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **16** No. 1, pp. 14-27
February 2007

Incremental Unknowns Method for Solving Three-Dimensional Convection-Diffusion Equations†

Lunji Song* and Yujiang Wu

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China.

Received June 2005; Accepted (in revised version) January 19, 2006

Abstract. We use the incremental unknowns method in conjunction with the iterative methods to approximate the solution of the nonsymmetric and positive-definite linear systems generated from a multilevel discretization of three-dimensional convection-diffusion equations. The condition numbers of incremental unknowns matrices associated with the convection-diffusion equations and the number of iterations needed to attain an acceptable accuracy are estimated. Numerical results are presented with two-level approximations, which demonstrate that the incremental unknowns method when combined with some iterative methods is very efficient.

Key words: Incremental unknowns; finite difference; convection-diffusion equation.

AMS subject classifications: 65M06, 65M60

1 Introduction

The incremental unknowns method was first introduced by Temam [1] to study long time integration of dissipative evolutionary equations when finite difference approximations are used. Incremental unknowns of different types have been proposed as a means to develop linear elliptic problem and nonlinear evolutionary equations. Garcia [2] studied the algebraic framework appropriate to one and two dimensional linear partial differential equations when several levels of discretization are considered. The hierarchical ordering of the nodal unknowns lead to a linear system $(A)(x) = (b)$, which can be written, with the use of the incremental unknowns ordered in the same way, as the equivalent system $[A][x] = [b]$, where $[A] = S^T(A)S$ and $[b] = S^T(b)$. Here, S stands for the transfer matrix from the incremental unknowns $[x]$ to the nodal unknowns (x), i.e., $(x) = S[x]$. Hereafter, we always solve the linear system $[A][x] = [b]$, instead of the linear system $(A)(x) = (b)$, with the use of the following iterative methods: the Conjugate Gradient method when $[A]$ is a symmetric and positive-definite matrix, the iterative methods such as MR,

*Correspondence to: Lunji Song, School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, China. Email: `song@lzu.edu.cn`
†This work was supported by the Foundation of Gansu Natural Science (3ZS041-A25-011).

GCR, Orthomin(k) [7], HSS and PSS [13, 14] when $[A]$ is a real nonsymmetric and positive-definite matrix, and the Bi-CGSTAB method (see [5]) when $[A]$ is a nonsingular matrix. By specializing the PSS method to block triangular (or triangular) and skew-Hermitian splittings (BTSS or TSS), the PSS method naturally leads to a BTSS or TSS iteration method, so the BTSS (or TSS) method is a special case of the PSS method.

The article is organized as follows. In Section 2, we introduce the convection-diffusion equations. In Section 3, we report six iterative methods, i.e., MR, GCR, Orthomin(k), Bi-CGSTAB, HSS, BTSS, for computing a nonsymmetric linear system. In Section 4, using the construction of transfer matrix S (see [11]) and the utiliztion of space decomposition, we analyze some properties of incremental unknowns and the transfer matrix. In Section 5, we apply the incremental unknowns method to solve the convection-diffusion equations and estimate condition numbers of the incremental unknown matrices. Moreover, it will be demonstrated that at most k iterations are needed to obtain an acceptable solution with MR, GCR, Orthomin(k). Numerical results with two-level approximations are presented and analyzed in Section 6.

2 Convection-diffusion equations

We consider the three-dimensional convection-diffusion equations

$$\begin{cases} Lu = -\nabla \cdot (\nabla u + bu) = f, & in\ \Omega, \\ u(x) = 0, & on\ \partial\Omega, \end{cases}$$

where $b > 0,\ x = (x_1,\ x_2\ ,x_3) \in \mathbb{R}^3$. When the central finite difference is used for the spatial multi-level discretization, we have the problem of approximating the solutions of large sparse systems of linear equations

$$AU = g, \tag{1}$$

where the vectors $U,\ g \in \mathbb{R}^N,\ A \in \mathbb{R}^{N\times N}$ is a nonsymmetric and positive-definite matrix of order N. The dimension N is the number of nodal points and U is the vector corresponding to the nodal values of the unknown function. In order to approximate the solution of (1), several iterative methods, such as Minimum Residual (MR), Generalized Conjugate Residual (GCR), Orthomin(k), Bi-CGSTAB methods, Hermitian and Skew-hermitian Splitting method (HSS) and Block Triangular and Skew-hermitian Splitting method (BTSS), can be considered.

We introduce incremental unknowns (IU) method to a linear system (see, [2, 3]). Let $\overline{U} \in \mathbb{R}^N$ be the $(l+1)$-level IU vector $(l > 0)$ of the form

$$\overline{U} = \begin{pmatrix} Y_0 \\ Z^l \end{pmatrix}, \quad Z^l = \begin{pmatrix} Z_1 \\ \vdots \\ Z_l \end{pmatrix},$$

where

(i). Y_0 is the properly ordered set of the approximate nodal values of u at the coarsest grid points;

(ii). Z_l is the properly ordered set of the incremental unknowns at the level l, which is the increment of u to the average of the values at the neighboring coarse grid points.

Moreover, let S be the transfer matrix from $\overline{U}$ to U

$$U = S\overline{U},$$