The Equivalence of Ishikawa-Mann and Multistep Iterations in Banach Space†

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Abstract. Let $E$ be a real Banach space and $T$ be a continuous $\Phi$–strongly accretive operator. By using a new analytical method, it is proved that the convergence of Mann, Ishikawa and three-step iterations are equivalent to the convergence of multistep iteration. The results of this paper extend the results of Rhoades and Soltuz in some aspects.

Key words: $\Phi$–strongly accretive operator; $\Phi$–strongly pseudocontractive operator; continuous; Mann iteration; Ishikawa iteration; multistep iteration.

AMS subject classifications: 47H09, 47H10

1 Introduction

Let $E$ denote an arbitrary real Banach space and $E^*$ denote the dual space of $E$. The duality map $J: E \rightarrow 2^{E^*}$ is defined by

$$Jx := \{u^* \in E^*: \langle x, u^* \rangle = \|x\|^2; \|u^*\| = \|x\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing between elements of $E$ and $E^*$. We first recall and define some concepts as follows:

**Definition 1.1.** Let $K$ be a nonempty subset of $E$ and let $T: K \rightarrow E$ be an operator.

(i). $T$ is said to be accretive if, for $\forall x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq 0. \quad (1)$$

(ii). $T$ is said to be strongly accretive if, for $\forall x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$\langle Tx - Ty, j(x - y) \rangle \geq k\|x - y\|^2, \quad (2)$$

where $k > 0$ is a constant. Without loss of generality, we assume that $k \in (0, 1)$.
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(iii). $T$ is said to be $\Phi-$strongly accretive if, for $\forall x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that

$$
(Tx - Ty, j(x - y)) \geq \Phi(||x - y||)||x - y||,
$$

(3)

where $\Phi: [0, \infty) \to [0, \infty)$ is a function for which $\Phi(0) = 0$, $\Phi(r) > 0$ for all $r > 0$, $\lim \inf_{r \to \infty} \Phi(r) = 0$ and the function $h(r) = r\Phi(r)$ is nondecreasing on $[0, \infty)$.

If $I$ denotes the identity operator, it follows from inequalities (1)-(3) that $T$ is pseudocontractive (respectively, strongly pseudocontractive, $\Phi-$strongly pseudocontractive) if and only if $(I - T)$ is an accretive (respectively, strongly accretive, $\Phi-$strongly accretive). It is shown in [1] that the class of single-valued strongly pseudocontractive operators is a proper subclass of the class of single-valued $\Phi-$strongly pseudocontractive operators. The classes of single-valued operators have been studied by many authors (see, for example [1]-[13]).

Now, we state concepts which will be needed in the sequel.

(a). The iteration (see [9])

$$
\begin{align*}
  x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n Ty_n^1, \\
  y_n^1 &= (1 - \beta_n^1)x_n + \beta_n^1 Tx_n, & n = 0, 1, 2, \ldots
\end{align*}
$$

(4)

is called the Ishikawa iteration sequence, where $\{\alpha_n\}, \{\beta_n^1\}$ are real sequences in $[0, 1]$ satisfying some appropriate conditions.

(b). In particular, if $\beta_n^1 = 0$ for $n \geq 0$, the sequence $\{x_n\}$ defined by

$$
x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, & n = 0, 1, 2, \ldots
$$

(5)

is called the Mann iteration (see [10]).

(c). In [11], Noor introduced the three-step procedure

$$
\begin{align*}
  x_{n+1} &= (1 - \alpha_n)x_n + Ty_n^1, \\
  y_n^1 &= (1 - \beta_n^1)x_n + \beta_n^1 Ty_n^2, \\
  y_n^2 &= (1 - \beta_n^2)x_n + \beta_n^2 Tx_n, & n = 0, 1, 2, \ldots,
\end{align*}
$$

(6)

where $\{\alpha_n\}, \{\beta_n^1\}, \{\beta_n^2\}$ are real sequences in $[0, 1]$ satisfying some appropriate conditions.

(d). In [13], Rhoades and Soltuz introduced the multi-step procedure

$$
\begin{align*}
  x_{n+1} &= (1 - \alpha_n)x_n + Ty_n^1, \\
  y_n^1 &= (1 - \beta_n^i)x_n + \beta_n^i Ty_n^{i+1}, & i = 1, \ldots, p - 2, \\
  y_n^{p-1} &= (1 - \beta_n^{p-1})x_n + \beta_n^{p-1} Tx_n, & n = 0, 1, 2, \ldots,
\end{align*}
$$

(7)

where $p \geq 2$ is a fixed order, $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that for all $n \in \mathbb{N}$

$$
\lim_{n \to \infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.
$$

(8)

Moreover, for all $n \in \mathbb{N}$

$$
\{\beta_n^i\} \subset [0, 1), \quad 1 \leq i \leq p - 1, \quad \lim_{n \to \infty} \beta_n^1 = 0.
$$

(9)