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Orthogonal Matrix-Valued Wavelet Packets†

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Abstract. In this paper, we introduce matrix-valued multiresolution analysis and matrix-valued wavelet packets. A procedure for the construction of the orthogonal matrix-valued wavelet packets is presented. The properties of the matrix-valued wavelet packets are investigated. In particular, a new orthonormal basis of $L^2(\mathbb{R},\mathbb{C}^{s\times s})$ is obtained from the matrix-valued wavelet packets.

Key words: Matrix-valued multiresolution analysis; matrix-valued scaling functions; matrix-valued wavelet packets; refinement equation.

AMS subject classifications: 42C40, 65T60

1 Introduction

Wavelet packets, due to their nice characteristics, have been applied to signal processing [1], image compression [2], integral equations [3] and so on. Coifman and Meyer [4] firstly introduced the concept of orthogonal wavelet packets. The introduction for biorthogonal wavelet packets was attributable to Cohen and Daubechies [5]. Furthermore, Yang and Cheng [6] constructed a-scale orthogonal multiwavelet packets which are more flexible in applications. Recently, the multiwavelets have become the focus of active research both in theory and application, such as signal processing [7], mainly because of their ability to offer properties like orthogonality and symmetry simultaneously. The matrix-valued wavelets are a class of generalized multiwavelets. Xia and Suter [8] introduced the concept of the matrix-valued wavelets and investigated its construction. Moreover, they showed that multiwavelets can be generated from the component functions of matrix-valued wavelets. However, the multiwavelets and matrix-valued wavelets are different in the following sense. For example, prefiltering is usually required for discrete multiwavelet transforms [9] but not necessary for discrete matrix-valued wavelet transforms. A typical example of such matrix-valued signals is video images. Hence, studying the matrix-valued wavelets is useful in representations of signals. It is necessary to extend the concept of orthogonal wavelet packets to the case of orthogonal matrix-valued wavelets. Based on an observation in

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[8] and some ideas from [5,6], we will give the definition for 3-scale orthogonal matrix-valued wavelet packets and investigate the properties of the orthogonal matrix-valued wavelet packets by using matrix theory and integral transform.

Throughout the paper, we use the following notations. Let $\mathbb{R}$ and $\mathbb{C}$ be sets of all real and complex numbers, respectively. $\mathbb{Z}$ stands for all integers. Set $s \in \mathbb{Z}$, $s \geq 2$, and $\mathbb{Z}_+ = \{z : z \geq 0, z \in \mathbb{Z}\}$. By $\mathbf{I}_s$ and $\mathbf{O}$, we denote the $s \times s$ identity matrix and zero matrix, respectively.

$$L^2(\mathbb{R}, \mathbb{C}^{s\times s}) := \left\{ \hbar(t) := \begin{pmatrix} h_{11}(t) & h_{12}(t) & \cdots & h_{1\,s}(t) \\ h_{21}(t) & h_{22}(t) & \cdots & h_{2\,s}(t) \\ \cdots & \cdots & \cdots & \cdots \\ h_{s1}(t) & h_{s2}(t) & \cdots & h_{s\,s}(t) \end{pmatrix} : \begin{array}{c} t \in \mathbb{R},\ h_{k\,l}(t) \in L^2(\mathbb{R}), \\ k, l = 1, 2, \cdots, s \end{array} \right\}$$

The signal space $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called a matrix-valued function space. Examples of matrix-valued signals are video images where $h_{k\,l}(t)$ is the pixel on the kth row and the lth column at time t.

For each $\hbar \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, $||\hbar||$ represents the norm of operator $\hbar$ as

$$||\,\hbar|| := \left(\sum_{k,\,l=1}^{s} \int_{\mathbb{R}} |h_{k,\,l}(t)|^2 dt \right)^{1/2}. \tag{1}$$

which is the norm used in this paper for the matrix-valued function spaces $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$.

For $\hbar \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, its integration $\int_{\mathbb{R}} \hbar(t)dt$ is defined as $\int_{\mathbb{R}} \hbar(t)dt := (\int_{\mathbb{R}} h_{k,l}(t)\,dt\,)_{k,l=1}^{s}$, where $\hbar(t)$ is the matrix-valued functions $(\,h_{k,\,l}(t)\,)_{k,\,l=1}^{s}$ to be defined below. The Fourier transform of $\hbar(t)$ is defined by $\widehat{\hbar}(\omega) := \int_{\mathbb{R}} \hbar(t) \exp\{-i\omega t\}\,dt,\ \omega \in \mathbb{R}$.

For two matrix-valued functions $\hbar,\ \Upsilon \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$, their *symbol inner product* is defined by $[\,\hbar, \Upsilon\,] := \int_{\mathbb{R}} \hbar(t)\Upsilon(t)^*\,dt$. Here and afterwards, * means the transpose and the complex conjugate.

Definition 1.1. A sequence $\{\hbar_k(t)\}_{k\in\mathbb{Z}} \subset \mathbf{X} \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called an orthonormal set in $\mathbf{X}$, if it satisfies

$$[\,\hbar_k, \hbar_l\,] = \delta_{k,\,l}\mathbf{I}_s, \qquad k,\, l \in \mathbb{Z} \tag{2}$$

where $\delta_{k,\,l}$ is the Kronecker symbol, i.e., $\delta_{k,\,l} = 1$ as $k = l$ and $\delta_{k,\,l} = 0$ otherwise.

Definition 1.2. A matrix-valued function $\hbar(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is said to be orthonormal, if $\{\hbar(t-k)\}_{k\in\mathbb{Z}}$ is an orthonormal set.

Definition 1.3. A sequence of matrix-valued functions $\{\hbar_k(t)\}_{k\in\mathbb{Z}} \subset \mathbf{X} \subset L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ is called an orthonormal basis of $\mathbf{X}$ if it satisfies (2) and for any $\Upsilon(t) \in \mathbf{X}$, there exists a unique matrix sequence $\{P_k\}_{k\in\mathbb{Z}}$ such that $\Upsilon(t) = \sum_{k\in\mathbb{Z}} P_k \hbar_k(t),\ \ t \in \mathbb{R}$.

This paper is organized as follows. In Section 2, we briefly recall the concepts relevant to the matrix-valued multiresolution analysis. In Section 3, we give our main result, and some properties of the matrix-valued wavelet packets.

2 Matrix-valued multiresolution analysis and wavelets

We begin with the generic setting of a matrix-valued multiresolution analysis of $L^2(\mathbb{R}, \mathbb{C}^{s\times s})$. Let $\mathbf{S}\,(t) \in L^2(\mathbb{R}, \mathbb{C}^{s\times s})$ satisfy the following refinement equation:

$$\mathbf{S}\,(t) = 3 \cdot \sum_{k\in\mathbb{Z}} A_k\, \mathbf{S}(3\,t - k), \tag{3}$$