NUMERICAL MATHEMATICS
A Journal of Chinese Universities (English Series)

Vol. **15** No. 3, pp. 217-226
August 2006

Least-Squares Mirrorsymmetric Solution for Matrix Equations ($\boldsymbol{AX = B, XC = D}$)

Fanliang Li[1,2,*], Xiyan Hu[2] and Lei Zhang[2]

[1] *Department of Mathematics and Computational Science, Hunan City College, Yi Yang 413000, China.*
[2] *College of Mathematics and Econometrics, Hunan University, Changsha 410082, China.*

Received June 9, 2005; Accepted (in revised version) October 24, 2005

Abstract. In this paper, least-squares mirrorsymmetric solution for matrix equations ($AX = B$, $XC = D$) and its optimal approximation is considered. With special expression of mirrorsymmetric matrices, a general representation of solution for the least-squares problem is obtained. In addition, the optimal approximate solution and some algorithms to obtain the optimal approximation are provided.

Key words: Mirrorsymmetric matrix; least-squares solution; optimal approximation.

AMS subject classifications: 15A06; 15A57

1 Introduction

Matrix equations ($AX = B, XC = D$) is a class of important matrix equations. The problem of the solution of the matrix equations ($AX = B, XC = D$) arise in engineering and in some special matrix inverse problems [1-3]. Many authors have been devoted to the study of this problem, and a series of useful results have been obtained. For example, Mitra [4] gave the common solution of minimum possible rank by using generalized inverse of matrix. Chu [5], Mitra [6] presented the necessary and sufficient conditions for the solvability and general solution by using the singular value decomposition (SVD) and generalized inverse of matrix, respectively. When the solution matrix X is constrained and the matrix equations are not consistent, it is necessary to study the least-squares constrained solution of the corresponding matrix equations.

The purpose of this paper is to discuss the least-squares mirrorsymmetric solution of the matrix equations ($AX = B, XC = D$) by using the special structure of mirrorsymmetric matrices. The background for introducing the definition of mirrorsymmetric matrices is to study odd/even-mode decomposition of symmetric multiconductor transmission lines (MTL)[7].

We now introduce some notation. Let $R^{n\times m}$ be the set of all $n \times m$ real matrices; $R_r^{n\times m}$ be the set of all matrices in $R^{n\times m}$ with rank r, $R(A)$ denote the rank of A; $OR^{n\times n}$ denote

*Correspondence to: Fanliang Li, Department of Mathematics and Computational Science, Hunan City College, Yi Yang 413000, China/College of Mathematics and Econometrics, Hunan University, Changsha 410082, China. E-mail: `lfl302@tom.com`

the set of all $n \times n$ orthogonal matrices; The identity matrix of order n by I_n; A^T and A^+ be the transpose and the Moore-Penrose generalized inverse of A, respectively. For $A = (a_{ij})$, $B = (b_{ij}) \in R^{n\times m}$, $A * B = (a_{ij}b_{ij})$ denote the Hadamard product of matrices A and B; $(A, B) = trB^TA$ denote the inner product of matrices A and B. The induced matrix norm is Frobenius norm, i.e $||A|| = (A, A)^{\frac{1}{2}} = (tr(A^TA))^{\frac{1}{2}}$.

Definition 1.1. The (k, p)-mirror matrix $W_{(k,p)}$ is defined by

$$W_{(k,p)} = \begin{pmatrix} & & J_k \\ & I_p & \\ J_k & & \end{pmatrix}, \tag{1}$$

where J_k is the k-square backward identity matrix with ones along the secondary diagonal and zero elsewhere.

The dimension of the (k, p)-mirror matrix is $n = 2k+p$, where $k \geq 1, p \geq 0$. The (k, p)-mirror matrix $W_{(k,p)}$ is orthogonal and symmetric, i.e. $W^{-1} = W^T = W$. When $p = 0$ or 1, mirror matrix $W_{(k,p)}$ is backward identity matrix J_n.

Definition 1.2. A matrix $A \in R^{(2k+p)\times(2k+p)}$ is called the (k, p)-mirrorsymmetric matrix if and only if

$$A = W_{(k,p)}AW_{(k,p)}. \tag{2}$$

We denote the set of all (k, p)-mirrorsymmetric matrices by $MS_{(k,p)}$. A matrix $A \in R^{(2k+p)\times(2k+p)}$ is called the (k, p)-mirrorskewsymmetric matrix if and only if

$$A = -W_{(k,p)}AW_{(k,p)}. \tag{3}$$

We denote the set of all (k, p)-mirrorskewsymmetric matrices by $MSS_{(k,p)}$.

From Definition 1.2, it is easy to see that the $(k, 1)$-mirrorsymmetric matrices and the $(k, 0)$-mirrorsymmetric matrices are centrosymmetric matrices. That is to say, all centrosymmetric matrices are the special cases of mirrorsymmetric matrices, i.e. when $p = 0$ or 1, mirror matrix $W_{(k,p)}$ is the backward identity matrix J_n. Then (2) becomes $A = J_nAJ_n$,which is the definition of centrosymmetric matrices [8].

We study the following problems in this paper.

Problem I. Given $A \in R^{h\times(2k+p)}, B \in R^{h\times(2k+p)}, C \in R^{(2k+p)\times l}, D \in R^{(2k+p)\times l}$, finding $X \in MS_{(k,p)}$ such that $f(X) \triangleq ||AX - B||^2 + ||XC - D||^2 = \min$.

Problem II. Given $X^* \in R^{(2k+p)\times(2k+p)}$ finding $\tilde{X} \in S_E$ such that

$$\|X^* - \tilde{X}\| = \min_{\forall X \in S_E} \|X^* - X\|,$$

where S_E is the solution set of Problem I.

The paper is organized as follows. In Section 2, we at first discuss the structure of mirrorsymmetric matrices. Then with the special structure of mirrorsymmetric matrices and the singular value decomposition (SVD) of matrix, we obtain the solution set of Problem I. In Section 3, the unique approximation solution of Problem II is presented by applying the decomposition of space. Finally, some algorithms to obtain the optimal approximate solution and numerical experiment to illustrate the results obtained in this paper correction are provided.