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Construction of Real Band Anti-Symmetric Matrices from Spectral Data

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Abstract. In this paper, we describe how to construct a real anti-symmetric $(2p-1)$-band matrix with prescribed eigenvalues in its p leading principal submatrices. This is done in two steps. First, an anti-symmetric matrix B is constructed with the specified spectral data but not necessary a band matrix. Then B is transformed by Householder transformations to a $(2p-1)$-band matrix with the prescribed eigenvalues. An algorithm is presented. Numerical results are presented to demonstrate that the proposed method is effective.

Key words: anti-symmetric; eigenvalues; inverse problem.

AMS subject classifications: 65F10, 15A09

1 Introduction

This work deals with inverse eigenvalue problems for real banded anti-symmetric matrices. The solution of inverse eigenvalue problems is currently attracting a great interest due to their importance in many applications. In particular, real banded matrices play an important role in areas as applied mechanics [1,2], structure design [3], circuit theory and inverse Sturm-Liouville problem [4].

Let $p, n \in N, 0 < p \leq n$ and $\{\lambda_j^{(k)}\}_{j=1}^{k} (k = n-p+1, \cdots, n)$ be a set of real numbers with

$$\lambda_j^{(k)} = -\lambda_{k-j+1}^{(k)}, \ j = 1, \cdots, k; k = n-p+1, \cdots, n. \tag{1}$$

$$\lambda_j^{(k)} \leq \lambda_j^{(k-1)} \leq \lambda_{j+1}^{(k)}, \ j = 1, \cdots, k-1; k = n-p+2, \cdots, n. \tag{2}$$

The problem is to determine a real anti-symmetric $n \times n$ matrix A with eigenvalues $\{\lambda_j^{(k)} i\}_{j=1}^{k}$ $(i^2 = -1)$ in the leading $k \times k$ principal submatrix of $A (k = n-p+1, \cdots, n)$ and $a_{st} = 0$ for $|s-t| \geq p$. In this paper a matrix A is called real anti-symmetric if $A \in R^{n\times n}, A^T = -A$. A similar problem with symmetric matrices has been studied in many papers, (see [5–10]). For anti-symmetric matrices, the case $p = 2$ has been studied by He Chengcai [11], but the complex numbers were used there, so that the computation is rather complicated .

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In Section 2 the eigen-properties of real anti-symmetric matrices were studied. In Section 3 an anti-symmetric matrix B is constructed where B has the specified spectral data, but it is not necessary a banded matrix. In Section 4 B is transformed to a $(2p-1)$-band matrix with the prescribed spectra. In Section 5, an algorithm is presented with numerical examples which show that the method is effective.

2 Some properties of real anti-symmetric matrices

In order to prove our main results, let us first investigate the eigen- properties of real anti-symmetric matrices. Some of them are well known so the proof is omitted.

Let A be a real anti-symmetric $n \times n$ matrix, i.e. $A \in R^{n\times n}, A^T = -A$. Then $-iA \in C^{n\times n}, (-iA)^H = iA^T = -iA$, hence $-iA$ is Hermitian and its eigenvalues are real. Let $\{\lambda_j^{(k)}\}_{j=1}^k$ be the eigenvalues of the $k \times k$ leading principal submatrix of $-iA (k = 1, \cdots, n)$ satisfying

$$\lambda_1^{(k)} \le \lambda_2^{(k)} \le \cdots \le \lambda_k^{(k)}. \tag{3}$$

According to Cauchy interlacing theorem, we have

$$\lambda_j^{(k)} \le \lambda_j^{(k-1)} \le \lambda_{j+1}^{(k)},\ j = 1, \cdots, k-1; k = 2, \cdots, n. \tag{4}$$

Noting that $A = i(-iA)$, we assert that $\{\lambda_j^{(k)} i\}_{j=1}^k$ be the eigenvalues of $k\times k$ leading submatrix of A and (4) hold. Furthermore, because $\{\lambda_j^{(k)} i\}_{j=1}^k$ are roots of a polynomial with real coefficients, so

$$\lambda_j^{(k)} = -\lambda_{k-j+1}^{(k)},\ j = 1, \cdots, k; k = 1, \cdots, n. \tag{5}$$

Lemma 2.1. *The eigenvalues of real anti-symmetric $n\times n$ matrix A are either zeroes or conjugate imaginaries. Let $\{\lambda_j^{(k)} i\}_{j=1}^k$ be the eigenvalues of the $k \times k$ leading submatrices of A satisfying* (3)*, then* (4) *and* (5) *hold.*

Lemma 2.2. *[12, 2.5.14] $A \in R^{n\times n}$ is anti-symmetric if and only if there exist an orthogonal matrix $U \in R^{n\times n}$ such that*

$$T = U^T A U = \begin{bmatrix} 0 & & & & & & & & & & \\ & \ddots & & & & & & & & & \\ & & 0 & & & & & & & & \\ & & & 0 & \beta_1 & & & & & & \\ & & & -\beta_1 & 0 & & & & & & \\ & & & & & 0 & \beta_2 & & & & \\ & & & & & -\beta_2 & 0 & & & & \\ & & & & & & & \ddots & & & \\ & & & & & & & & 0 & \beta_r \\ & & & & & & & & -\beta_r & 0 \end{bmatrix} \tag{6}$$

and $\pm\beta_1 i, \cdots, \pm\beta_r i$ are all non-real eigenvalues of A. In this paper, T is referred to as the normal canonical form of A if $\beta_1 \le \beta_2 \le \cdots \le \beta_r$ in (6).

Remark 2.1. The orthogonal matrix U can be chosen as follow: the first $n-2r$ columns are the orthonormal eigenvectors corresponding to zero eigenvalues of A, the remaining columns are the orthonormal imagine part and real part of the eigenvectors corresponding to eigenvalues $\beta_1 i, \cdots, \beta_r i$ respectively. The orthonormalization is needed when there are multiple eigenvalues.