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A NOTE ON SENSITIVITY ANALYSIS OF SHAPE-FROM-MOMENTS*

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Abstract *The shape-from-moments problem is to reconstruct a planar polygon from a set of its complex moments. To reconstruct a polygon means to estimate the vertices and the ordering of the vertices. We notice that some coefficients are very important in finding out the ordering of the vertices. We introduce sensitive factors for the coefficients and use it to analyze sensitivity. These factors are also useful for the sensitivity of the vertices.*

Key words *shape-from-moments, sensitivity analysis.*

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1 Introduction

Let $z_1,z_2,\cdots,z_N$ be the vertices of a polygon P in the complex plane. Suppose that the vertices z_j of P are arranged in the clockwise direction, and denote $z_0 = z_N$, $z_{N+1} = z_1$. From Davis Theorem [1], there exist constant coefficients $a_1,a_2,\dots,a_N$, such that

$$k(k-1)\iint_P z^{k-2}\,\mathrm{d}x\mathrm{d}y = \sum_{j=1}^{N} a_j z_j^k. \tag{1}$$

Let θ_j be the angle of the vector $z_{j-1} - z_j$, then

$$a_j = \sin(\theta_{j-1} - \theta_j)e^{-i(\theta_{j-1}+\theta_j)}. \tag{2}$$

The complex moments τ_k of the polygon are then defined as

$$\tau_k = \sum_{j=1}^{N} a_j z_j^k. \tag{3}$$

The reconstruction problem of the polygon (shape-from-moments) is: For given $M+1$ complex moments $\{\tau_k\}_{k=0}^{M}$, reconstruct the polygon using these moments. This problem is widespread in both pure and applied mathematics, such as computed tomography, probability

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and statistics[9], signal processing [10]etc. Milanfar and Karl[5] established a set of results showing that the vertices of a simply connected polygon in the complex plane are determined by a finite number of its moments. The techniques used are from numerical linear algebra, such as least-square problems, or generalized eigenvalue problems, we refer the readers to [5,2] for details.

To reconstruct a polygon includes two steps: 1, estimate the vertices $\{z_k\}_{k=0}^N$; 2, find out the ordering of the vertices. In step 1, one alternative relation to (3) leads to the least-square Prony's method [7,5]. Pencil based methods [2] transform the shape-from-moments problem to a generalized eigenvalue problem: $T_1u = zT_0u$, generalized eigenvalues z_j are the vertices. GPOF(generalized pencil method) is given by Hua and Sarkar [3,4]. One question here is: For noised given moments, or in case of finite precision computation, how to check the correctness of the computed vertices? If pencil method is used, the sensitivity of the vertices with respect to perturbations in the moments can be computed from the eigenvalue sensitivity analysis, which uses classical results in numerical linear algebra, see, e.g. [2]. In step 2, after the vertices z_j have been estimated, there are several techniques for computing the coefficients a_j. Until now the simplest technique is to use (3), which leads to a linear least squares problem. Then we can find out the ordering of the vertices directly from equation (2) in some simple cases, for example, convex polygons. We refer the readers to [6] for finding the order of the vertices from a_j. In this paper, we analyze the sensitivity of coefficients a_j. For each j, we give a sensitivity factor. If the sensitivity factor is large, then computed a_j may be not accuracy. Numerical results show that the results are also true for computed vertices.

2 Sensitivity Analysis

Sensitivity analysis on pencil methods is given in [2], by using ϵ-perturbation of the matrix pencil. All eigenvectors should be computed for sensitivity analysis. But the eigenvectors are additional computing since they are not used in the computation for vertices z_j and coefficients a_j. We have to mention that in some cases pencil methods are very sensitive and does not show any promise [7].

From section 1 we know that coefficients a_j are very important in finding out the ordering of the vertices. If the vertices are already computed, then the coefficients a_j can be computed from the original definition of τ_k.

Let matrix V^T be the Vandermonde matrix of z_j, i.e.

$$V = \begin{pmatrix} z_1^0 & z_2^0 & \cdots & z_N^0 \\ z_1^1 & z_2^1 & \cdots & z_N^1 \\ z_1^2 & z_2^2 & \cdots & z_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^M & z_2^M & \cdots & z_N^M \end{pmatrix}, \tag{4}$$

τ and a be column vector of τ_k and a_j, then (3) can be rewritten in matrix form

$$Va = \tau. \tag{5}$$

If the moments and the vertices are exact, then (5) is an overdetermined equation. Normally it is a least square problem because of noise in the given moments and roundoff errors in