THE MONOTONICITY OF CONVERGENCE RATE FOR MGS METHODS

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Abstract  In this paper we prove that the asymptotic rate of convergence of the modified Gauss-Seidel method of a non-singular $M$-matrix is a monotonic function for precondition parameters $0 \leq \alpha_i \leq \frac{1}{2}$, $(i = 1, 2, \cdots, n - 1)$.

Key words  Gauss-Seidel method, convergence rate, monotonicity.

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1 Introduction

Let $A$ be an $n \times n$ matrix with all diagonal entries 1, $-L$ and $-U$ be strictly lower and strictly upper triangular part of $A$, respectively. Then the Gauss-Seidel splitting of $A$ has the form that $A = (I - L) - U$, where $I$ is the identity matrix of order $n$. For the convenience of statement, we take the notations as follows:

$$V = \begin{bmatrix}
0 & -a_{12} & 0 & \cdots & 0 \\
0 & 0 & -a_{23} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & -a_{n-1,n}
\end{bmatrix}$$

and $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1})$, $D_\alpha = \text{diag}(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, 1)$, $S_\alpha = D_\alpha V$, $P_\alpha = I + S_\alpha$, $A_\alpha = P_\alpha A$. Briefly, we denote $D_c, S_c, P_c, A_c$, etc. for the case $\alpha_i (\forall i)$ all $c$, respectively.

Consider Gauss-Seidel splitting $A_\alpha \doteq E_\alpha - F_\alpha$. Observably, $A_0$ is the case of standard Gauss-Seidel splitting of $A$. Gunawardena et al [1] studied firstly the convergence for $A_1$ (the Modified Gauss-Seidel method). And then Kohno et al [2] extended to the general case for $0 \leq \alpha \leq 1$.

When $A$ is a non-singular $M$-matrix, the iterative matrix $T_\alpha = E_\alpha^{-1}F_\alpha$ of Gauss-Seidel splitting for $A_\alpha$ is non-negative, and has the spectral radius $\rho_\alpha = \rho(T_\alpha) < 1$. Gunawardena’s works [1] show that $\rho_1 \leq \rho_0$, and Li’s works [3] show that $\rho_\alpha \leq \rho_0$ for $0 \leq \alpha \leq 1$. In [4], Li shows that $\rho_\alpha \geq \rho_\beta$ for $0 \leq \alpha \leq \beta \leq \varepsilon$, where $\varepsilon$ is some vector only relative to matrix $A$ when $A$ is a diagonally dominant non-singular $M$-matrix, and conjectures that the above statement would be true for $0 \leq \alpha \leq \beta \leq 1$. That is, $\rho_\alpha$ would be a monotonic function when $0 \leq \alpha \leq 1$.

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In this paper, we show Li’s conjecture true for any non-singular $M$-matrix when $0 \leq \alpha \leq \xi$, (where $\xi \geq \frac{1}{2}$ and only relative to A) without the assumption that $A$ is diagonally dominant.

2 Some Facts and Lemmas

Throughout the rest of this paper, we always suppose that $A$ is a non-singular $M$-matrix, $0 \leq \alpha \leq \beta \leq 1$, and $M_{\varepsilon} = P_{\varepsilon}^{-1}E_{\varepsilon}$, $N_{\varepsilon} = P_{\varepsilon}^{-1}F_{\varepsilon}$, $x_{\varepsilon}$ is a non-negative eigenvector belonging to $\rho_{\varepsilon}$ of $T_{\varepsilon}$ (where $\varepsilon = \alpha, \beta$). By simple computing, following facts can be obtained:

$$N_{\varepsilon} = (U - S_{\varepsilon}) + S_{\varepsilon}^2 (I - S_{\varepsilon}^2)^{-1} (I - S_{\varepsilon}),$$  (2.1)

$$M_{\varepsilon}^{-1} N_{\varepsilon} = E_{\varepsilon}^{-1} F_{\varepsilon} = T_{\varepsilon} \geq 0,$$  (2.2)

$$F_{\varepsilon} \geq 0.$$  (2.3)

Some lemmas without proof are stated as follows, which can be easily followed from [4][5]:

**Lemma 2.1** If $\rho_{\varepsilon} > 0$, $Ax_{\varepsilon} = \frac{1 - \rho_{\varepsilon}}{\rho_{\varepsilon}} N_{\varepsilon}x_{\varepsilon}$, and $A_{\varepsilon}x_{\varepsilon} = \frac{1 - \beta_{\varepsilon}}{\beta_{\varepsilon}} F_{\varepsilon} x_{\varepsilon} \geq 0$.

**Lemma 2.2** $T_{\alpha} A^{-1} \geq T_{\beta} A^{-1}$.

**Lemma 2.3** $A_{\varepsilon} = P_{\varepsilon} A$ is a non-singular $M$-matrix.

3 Results

Now, we show our main theorem.

**Theorem 3.1** Let $A$ be a non-singular $M$-matrix, $0 \leq \alpha \leq \beta \leq \xi$, where

$$\xi_i = \frac{1}{1 + \sqrt{1 - a_{i,i+1}a_{i+1,i}}} \quad (0 \leq i < n).$$

Then $\rho_{\alpha} \geq \rho_{\beta}$.

**Proof** When $\rho_{\beta} = 0$, $\rho_{\alpha} \geq 0 = \rho_{\beta}$, it obvious.

Now suppose that $0 < \rho_{\beta} < 1$.

Let $q_i = \frac{1}{1 - \beta_{\alpha}a_{i,i+1}a_{i+1,i}}$, $Q_{\beta} = \text{diag}(q_1, \cdots, q_{n-1}, 1)$, $s_i = -\beta_{i}a_{i,i+1},$ $(0 \leq i < n)$. Because $A$ is a non-singular $M$-matrix, $1 > a_{i,i+1}a_{i+1,i}$. So, $\xi_i < 1 \leq q_i,$ $(0 \leq i < n).$ Then

$$(Q_{\beta}A_{\beta} - (I - S_{\beta}))_{i,i+1} = (Q_{\beta}A_{\beta} + S_{\beta})_{i,i+1} = q_i (1 - \beta_{i}) a_{i,i+1} + s_i$$

$$= \frac{(1 - \beta_{i}) a_{i,i+1}}{1 - \beta_{i} a_{i,i+1} a_{i+1,i}} - \beta_{i} a_{i,i+1} = q_i a_{i,i+1} \cdot (1 - 2\beta_{i} + \beta_{i}^2 a_{i,i+1} a_{i+1,i}).$$

If $a_{i,i+1}a_{i+1,i} = 0$, then $1 - 2\beta_{i} \geq 1 - 2\xi_{i} = 0$. So, $(I - S_{\beta})_{i,i+1} \geq (Q_{\beta}A_{\beta})_{i,i+1}$ because of $a_{i,i+1} \leq 0$. While $a_{i,i+1}a_{i+1,i} > 0$, we have that

$$1 - 2\beta_{i} + \beta_{i}^2 a_{i,i+1}a_{i+1,i} = \left(1 - \frac{\beta_{i}}{1 - \sqrt{1 - a_{i,i+1}a_{i+1,i}}} - \beta_{i}\right) \cdot \left(1 - \frac{\beta_{i}}{1 + \sqrt{1 - a_{i,i+1}a_{i+1,i}}} - \beta_{i}\right)$$

$$= \left(1 - \frac{\beta_{i}}{1 - \sqrt{1 - a_{i,i+1}a_{i+1,i}}} - \beta_{i}\right) \cdot (\xi_{i} - \beta_{i}).$$