English Series | NUMERICAL MATHEMATICS A Journal of Chinese Universities | Vol.14, No.2 May 2005

DERIVATIVES OF EIGENPAIRS OF SYMMETRIC QUADRATIC EIGENVALUE PROBLEM*

Xie Huiqing (解惠青) Dai Hua(戴 华)

Abstract *Derivatives of eigenvalues and eigenvectors with respect to parameters in symmetric quadratic eigenvalue problem are studied. The first and second order derivatives of eigenpairs are given. The derivatives are calculated in terms of the eigenvalues and eigenvectors of the quadratic eigenvalue problem, and the use of state space representation is avoided, hence the cost of computation is greatly reduced. The efficiency of the presented method is demonstrated by considering a spring-mass-damper system.*

Key words *quadratic eigenvalue problem, eigenvalue, eigenvector, derivative.*

AMS(2000)subject classifications $65F15, 15A18$

1 Introduction

Sensitivity analysis of eigenvalue problems is widely used in structural design[1], model modification[2,3], damage detection[4]. Computation of derivatives of eigenpairs has been studied by many authors in the past 40 years[1,5]. Rellich[6], Lancaster[7], Garg[8], Rudisill[9], Nelson[10], Sun[11], Andrew and Tan[12] studied derivatives of eigenvalues and eigenvectors of a matrix. Fox and Kapoor[13], Rogers[14], Andrew and Tan[15] treated derivatives of eigenpairs of matrix pencils. Nevertheless, the following quadratic eigenvalue problem arises in many applications[1,16,17].

$$\left(\lambda^2 M + \lambda C + K\right) u = 0, \tag{1}$$

where $M = M(p), C = C(p), K = K(p)$ are $n \times n$ real symmetric-matrix-valued functions depending on parameters $p = (p_1, \cdots, p_N)^T$, $M(p), C(p), K(p)$ are analytic in a neighborhood of $p^* \in R^N$, and $M(p)$ is invertible in a neighborhood of p^*. Zeng[18] proposed a method to calculate derivatives of eigenpairs of the quadratic eigenvalue problem, but this method used state space representation, and required great numerical efforts. Adhikari[19] obtained derivatives of

* he research was supported by the National Natural Science Foundation of China (No.10271055).
Received: May. 27, 2003.

eigenpairs in terms of eigenvalues and eigenvectors in n space, but the results are incorrect. Andrew, Chu and Lancaster[20] studied derivatives of eigenvalues and eigenvectors of general matrix-valued functions. However, little effort in the second order derivatives of eigenpairs of the quadratic eigenvalue problem can be found in the existing literature.

Calculation of derivatives of eigenpairs of the symmetric quadratic eigenvalue problem is investigated in n space. The expressions of the first and second order derivatives of eigenpairs are given, and a spring-mass-damper system is considered to show the efficiency of the presented method.

2 The first order derivatives of eigenpairs

Without loss of generality, it is assumed throughout that p^* is the origin of R^N.

Suppose all eigenvalues $\lambda_1, \cdots, \lambda_{2n}$ of (1) at the origin are distinct, $u_i \in C^n$ is an eigenvector corresponding to the eigenvalue λ_i. By Theorem 3.2 in [20], it follows that $u_i^T\left(2\lambda_i M + C\right) u_i \neq 0$. We may normalize the eigenvectors so that $u_i^T\left(2\lambda_i M(0) + C(0)\right) u_i = 1$. The following results are easily proved.

Theorem 1 Suppose that M, C, K be $n \times n$ real symmetric matrices, M is invertible. Let

$$A = \begin{bmatrix} -K & 0 \\ 0 & M \end{bmatrix}, \qquad B = \begin{bmatrix} C & M \\ M & 0 \end{bmatrix}, x^T = \left[u^T, \lambda u^T\right].$$

(i) The symmetric quadratic eigenvalue problem (1) is equivalent to the symmetric generalized eigenvalue problem

$$Ax = \lambda Bx, \tag{2}$$

(ii) Let $u \in C^n$ be an eigenvector corresponding to eigenvalue λ, and $u^T\left(2\lambda M + C\right) u = 1$, then

$$Ax = \lambda Bx, x^T A = \lambda x^T B, x^T Bx = 1,$$

(iii) If λ_1, λ_2 are distinct eigenvalues of (2), $x_1, x_2 \in C^{2n}$ are eigenvectors corresponding to λ_1, λ_2, respectively, then $x_1^T Bx_2 = 0$.

Let $X = [x_1, \cdots, x_{2n}]$, where

$$x_i^T = \left[u_i^T, \lambda_i u_i^T\right], i = 1, \cdots, 2n. \tag{3}$$

From Theorem 1, we have

$$A(0)\, x_i = \lambda_i B(0)\, x_i, x_i^T A(0) = \lambda_i x_i^T B(0),$$

$$X^T B(0)\, X = I_{2n}, X^T A(0)\, X = \Lambda = \operatorname{diag}\left(\lambda_1, \cdots, \lambda_{2n}\right). \tag{4}$$