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# A MPI PARALLEL PRECONDITIONED SPECTRAL ELEMENT METHOD FOR THE HELMHOLTZ EQUATION*

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**Abstract** *Spectral element method is well known as high-order method, and has potential better parallel feature as compared with low order methods. In this paper, a parallel preconditioned conjugate gradient iterative method is proposed to solving the spectral element approximation of the Helmholtz equation. The parallel algorithm is shown to have good performance as compared to non parallel cases, especially when the stiffness matrix is not memorized. A series of numerical experiments in one dimensional case is carried out to demonstrate the efficiency of the proposed method.*

**Key words** *Spectral element method, parallel computing, finite element preconditioner.*

**AMS(2000)subject classifications** 65N35, 65N55

## 1 Introduction

The spectral element method (SEM) is essentially a discretization method for the approximate solution of partial differential equations expressed in a weak form, based on high-order Lagrangian interpolants used in conjunction with particular quadrature rules. It combines the geometric flexibility of finite element techniques with rapid convergence rate of spectral schemes [7]. Due to these advantages, the spectral element method is a viable alternative to currently popular methods such as finite volumes and finite elements, if accurate solutions of regular problems are sought. Another benefit of the SEM is that it is convenient to be paralleled in the implementation. In practical applications, moderate number of elements and high order polynomial degree are used, consequently most computation is performed within, rather than between,

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elements. This feature makes the SEM great feasible to be implemented in parallel way.

Despite these nice features, the SEM has been thought to be more expensive in terms of the computational complexity and condition number of the stiffness matrix, as compared to the finite element methods with the same number of degree-of-freedom. There exists different approaches currently adopted in the SEM. The traditional approach in the spectral element community [7] has been to used a nodal tensorial expansion basis within structured subdomains (i.e., quadrilaterals or hexahedral elements). These bases are typically constructed from Lagrange polynomials through Gauss Lobatto Legendre quadrature points. In large-scale problems, long-range interactions between the basis elements within each substructure result in quite dense and expensive factorizations of the stiffness matrix, and the use of direct methods is hence not economical because of the large memory requirements [5]. In the past decade, iterative methods have been developed to solving the spectral element discretization problems of various equations. However, it has also been proven that the condition number of the SE stiffness matrix of the Laplacian operator is of order $O(KN^{d+1})$ [2], where $K$ is the element number, $N$ is the polynomial degree, $d$ is the spatial dimension. Naive iterative methods are not really efficient due to the fact that too many iterations are required to reach the convergence. For a long time, many preconditioners have been proposed to overcome this difficulty. Orszag [12] and Deville et al. [3] proposed the use of a finite difference and a finite element model, respectively, as preconditioners for the spectral matrix. The triangulation for this finite element method was based on the hexahedrals of the Gauss-Lobatto-Legendre (GLL) mesh of one element. A theoretical justification of this preconditioning is provided by Parter and Rothman in [10,8]. Extension of these ideas to the SEM was proposed by Fischer [4], who used overlapping Schwarz methods applied to the GLL finite element model. The generalization of the result of [8] to the multi-element case was given by Huang and Xu in [6].

In this paper we follow the above mentioned ideas, but consider a different way to implement iterative algorithm for the preconditioned spectral element method. Precisely, we will consider a parallel preconditioned conjugate gradient method for the spectral element stiffness matrix using the linear finite element method as the preconditioner. Our main aim is to carry out a series of numerical experiences to show that use of the parallel algorithm can significantly reduce the CPU time of solving the spectral element system of the Helmholtz equation.

## 2 Formulation of the problem

We consider the following one dimensional Helmholtz equation: Find a function $u$ defined in $\Omega = (a, b)$, such that

$$\begin{cases} -(pu')' + \lambda^2 u = f, \text{ in } \Omega, \\ u(a) = u(b) = 0, \end{cases} \tag{1}$$

where $\lambda$ is a real number, prime (') denotes differentiation with respect to $x$, $f(x), p(x)$ are functions defined over $\Omega$, which are assumed that there exists two positive constants $\tau_0$ and $\tau_\infty$