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Bivariate Blending Thiele-Werner's Osculatory Rational Interpolation

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Abstract

Both the expansive Newton's interpolating polynomial and the Thiele-Werner's interpolation are used to construct a kind of bivariate blending Thiele-Werner's osculatory rational interpolation. A recursive algorithm and its characteristic properties are given. An error estimation is obtained and a numerical example is illustrated.

Keywords: Bivariate blending interpolation; osculatory rational interpolation; Thiele-Werner's interpolation; error estimation.
Mathematics subject classification: 41A20, 65D05

1. Introduction

Given a set of real points $X_n = \{x_0, x_1, \cdots, x_n\} \subset [a, b] \subset R$, a function $f(x)$ defined on set $[a, b]$ can be approximated by the expansive Newton's interpolating polynomials [5]

$$N_n(x) = \sum_{i=0}^{n} f[x_0, x_0, \cdots, x_{i-1}, x_{i-1}, \overbrace{x_i, \cdots, x_i}^{l}] \prod_{v=0}^{i-1} (x - x_v)^2 (x - x_i)^{l-1}.$$

An efficient and reliable method of rational interpolation was proposed by Werner [1979] who considers generalized Thiele interpolants of the form [1]

$$R^{(0)}(x) = p_0(x) + \frac{\omega_0(x)}{p_1(x)} + \frac{\omega_1(x)}{p_2(x)} + \cdots + \frac{\omega_{t-1}(x)}{p_t(x)}.$$

In this formula, the polynomial

$$\omega_s(x) = \prod_{i=h_s}^{k_s} (x - x_i) \quad (s = 0, 1, \cdots, t-1)$$

vanishes at the (possibly reordered) interpolation points $x_{h_s}, x_{h_s+1}, \cdots, x_{k_s}$; $p_s(x)$ is a Newton interpolating polynomial which interpolates $f^{(s)}(x)$ on $x_{h_s}, x_{h_s+1}, \cdots, x_{k_s}$. The data for each stage of the iterative construction of the fraction are defined by

$$f^{(s+1)}(x_i) = \frac{\omega_s(x_i)}{f^{(s)}(x_i) - p_s(x_i)},$$

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for $s=0,1,\cdots,t-1$, $i=h_{s+1},h_{s+1}+1,\cdots,n$, $n=\sum_{s=0}^{t-1}(k_s-h_s+1)$.

For simplicity and also without loss of generality, we restrict ourselves to the case where bivariate problems are involved [4] .

2. The construction of bivariate blending Thiele-Werner's osculatory rational interpolation

We set $X_n=\{x_0,x_1,\cdots,x_n\}\subset[a,b]\subset R$, $Y_m=\{y_0,y_1,\cdots,y_m\}\subset[c,d]\subset R$, $\Pi_{x,y}^{n,m}=X_n\times Y_m$, and let a bivariate function $f(x,y)$ be defined on $D=[a,b]\times[c,d]$.

Now, we will construct a bivariate function $R(x,y)$ of the following form

$$R(x,y)=p_0(x,y)+\frac{(x-x_0)^2}{p_1(x,y)}\,{+}\,\frac{(x-x_1)^2}{p_2(x,y)}\,{+}\cdots{+}\,\frac{(x-x_{n-1})^2}{p_n(x,y)}, \tag{2.1}$$

such that

$$R(x_i,y_j)=f(x_i,y_j),\quad \frac{\partial}{\partial x}R(x_i,y_j)=f_x(x_i,y_j),\quad \frac{\partial}{\partial y}R(x_i,y_j)=f_y(x_i,y_j). \tag{2.2}$$

Suppose $\varphi_i(x,y)$ $(i=0,1,\cdots,n)$ are bivariate functions defined on $D=[a,b]\times[c,d]$. We give the following two definitions firstly.

Definition 2.1. [5] *For fixed x, let*

$$\begin{aligned}
&\varphi_i[x;y]=\varphi_i(x,y),\qquad \varphi_i[x;y_j,y_j]=\tfrac{\partial}{\partial y}\varphi_i(x,y_j),\\
&\varphi_i[x;y_p,y_q]=\tfrac{\varphi_i[x;y_p]-\varphi_i[x;y_q]}{y_p-y_q},\qquad (p\neq q),\\
&\varphi_i[x;y_0,y_0,\cdots,\overbrace{y_p,\cdots,y_p}^{h},\overbrace{y_k,\cdots,y_k}^{l}]\\
&=\frac{\varphi_i[x;y_0,y_0,\cdots,\overbrace{y_p,\cdots,y_p}^{h},\overbrace{y_k,\cdots,y_k}^{l-1}]-\varphi_i[x;y_0,y_0,\cdots,\overbrace{y_p,\cdots,y_p}^{h-1},\overbrace{y_k,\cdots,y_k}^{l}]}{y_p-y_k},
\end{aligned}$$

where $h=1,2$; $l=1,2$; $0\le p<k$ *and* p,k *are non-negative integers. Then* $\varphi_i[x;y_0,y_0,\cdots,\overbrace{y_p,\cdots,y_p}^{h},\overbrace{y_k,\cdots,y_k}^{l}]$ *is called the expansive divided difference of Newton-type of* $\varphi_i(x,y)$ *at point sets with* $\{x_0,\cdots,x_n\}\times\{y_0,\cdots,y_m\}$.

Definition 2.2. [5] *For fixed x, let*

$$\begin{aligned}
&\varphi_i[x,x;y]=\tfrac{\partial\varphi_i(x,y)}{\partial x},\\
&\varphi_i[x,x;y_p,y_q]=\tfrac{\varphi_i[x,x;y_q]-\varphi_i[x,x;y_p]}{y_q-y_p},\\
&\varphi_i[x,x;y_0,\cdots,y_p,y_k]=\tfrac{\varphi_i[x,x;y_0,\cdots,y_{p-1},y_k]-\varphi_i[x,x;y_0,\cdots,y_p]}{y_k-y_p},
\end{aligned}$$

where $0\le p<k$, *and* p,k *are non-negative integers. Then* $\varphi_i[x,x;y_0,\cdots,y_p,y_k]$ *is called the divided difference of Newton-type of* $\frac{\partial\varphi_i(x,y)}{\partial x}$ *at the point sets with* $\{x_0,\cdots,x_n\}\times\{y_0,\cdots,y_m\}$.

Secondly, we give our recursive algorithm. Let $\omega_j(y) = (y-y_0)(y-y_1)\cdots(y-y_j)$ for $j \geq 0$ and $\omega_{-1}(y) = 1$.

Algorithm 2.1

Step 1 Let $\varphi_0(x,y) = f(x,y), x \in [a,b]$,

$$p_0(x,y) = \sum_{j=0}^{m}\Big(\varphi_0[x_0;y_0,y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}^2(y) + \varphi_0[x_0;y_0,y_0,y_1,\cdots,y_j,y_j]$$
$$\omega_{j-1}(y)\omega_j(y)\Big) + (x-x_0)\left(\sum_{j=0}^{m}\varphi_0[x_0,x_0;y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}(y)\right).$$

Step 2 Let $\varphi_1(x,y) = \frac{(x-x_0)^2}{\varphi_0(x,y)-p_0(x,y)}$, $x \in [x_1,b]$,

$$p_1(x,y) = \sum_{j=0}^{m}\Big(\varphi_1[x_1;y_0,y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}^2(y) + \varphi_1[x_1;y_0,y_0,y_1,\cdots,y_j,y_j]$$
$$\omega_{j-1}(y)\omega_j(y)\Big) + (x-x_1)\left(\sum_{j=0}^{m}\varphi_1[x_1,x_1;y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}(y)\right).$$

$\vdots$

Step (s+1) Let

$$\varphi_s(x,y) = \frac{(x-x_{s-1})^2}{\varphi_{s-1}(x,y)-p_{s-1}(x,y)} = \frac{(x-x_{s-1})^2}{-p_{s-1}(x,y)+}\frac{(x-x_{s-2})^2}{-p_{s-2}(x,y)+}\cdots+\frac{(x-x_1)^2}{-p_1(x,y)}$$
$$+\frac{(x-x_0)^2}{\varphi_0(x,y)-p_0(x,y)}, \qquad x \in [x_s,b], \tag{2.3}$$
$$p_s(x,y) = \sum_{j=0}^{m}\Big(\varphi_s[x_s;y_0,y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}^2(y) + \varphi_s[x_s;y_0,y_0,y_1,\cdots,y_j,y_j]$$
$$\omega_{j-1}(y)\omega_j(y)\Big) + (x-x_s)\left(\sum_{j=0}^{m}\varphi_s[x_s,x_s;y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}(y)\right), \tag{2.4}$$

for $s = 2,3,\cdots,n-1$.

So we have completed the construction of bivariate function $R(x,y)$ which is defined in (2.1) and $p_0(x,y), p_1(x,y), \cdots, p_n(x,y)$ are defined by Algorithm 2.1. Next, we will prove two lemmas in order to show that the constructed function $R(x,y)$ satisfies the interpolation conditions in (2.2).

Lemma 2.1. *Let $p_i(x,y)$ and $\varphi_i(x,y)$ be defined by Algorithm* 2.1. *Then $p_i(x,y)$ and $\varphi_i(x,y)$ satisfy:*

$$p_i(x_i,y_j)=\varphi_i(x_i,y_j),\quad \frac{\partial}{\partial x}p_i(x_i,y_j)=\frac{\partial}{\partial x}\varphi_i(x_i,y_j),\quad \frac{\partial}{\partial y}p_i(x_i,y_j)=\frac{\partial}{\partial y}\varphi_i(x_i,y_j),$$
$$i=0,1,\cdots,n,\quad j=0,1,\cdots,m.$$

Proof. From (2.4), Definitions 2.1 and 2.2, we derive

$$\begin{aligned}p_i(x_i,y_j)=&\varphi_i[x_i;y_0]+\varphi_i[x_i;y_0,y_0]\omega_0(y_j)+\cdots+\varphi_i[x_i;y_0,y_0,\cdots,y_{j-1},y_{j-1}]\\ &\omega_{j-2}(y_j)\omega_{j-1}(y_j)+\varphi_i[x_i;y_0,y_0,\cdots,y_{j-1},y_j]\omega_{j-1}^2(y_j)\\ =&\varphi_i[x_i;y_j]=\varphi_i(x_i,y_j),\end{aligned}\tag{2.5}$$

$$\begin{aligned}\frac{\partial}{\partial x}p_i(x_i,y_j)=&\varphi_i[x_i,x_i;y_0]+\varphi_i[x_i,x_i;y_0,y_1]\omega_0(y_j)+\cdots\\ &+\varphi_i[x_i,x_i;y_0,y_1,\cdots,y_{j-1},y_j]\omega_{j-1}(y_j)\\ =&\varphi_i[x_i,x_i;y_j]=\frac{\partial}{\partial x}\varphi_i(x_i,y_j),\end{aligned}\tag{2.6}$$

$$\begin{aligned}\frac{\partial}{\partial y}p_i(x_i,y_j)=&\frac{d}{dy}\Bigg(\sum_{t=0}^{m}\Big(\varphi_i[x_i;y_0,y_0,y_1,\cdots,y_{t-1},y_t]\omega_{t-1}^2(y)\\ &+\varphi_i[x_i;y_0,y_0,\cdots,y_t,y_t]\omega_{t-1}(y)\omega_t(y)\Big)\Big)\Big|_{y=y_j}\\ =&\varphi_i[x_i;y_j,y_j]=\frac{\partial}{\partial y}\varphi_i(x_i,y_j).\end{aligned}\tag{2.7}$$

(2.5) and (2.7) can be proved by the properties of expansive Newton interpolating polynomials and (2.6) can be proved easily through the properties of Newton interpolating polynomials. ∎

Lemma 2.2. *For $0\le s\le n$, let*

$$R_s(x,y)=p_s(x,y)+\frac{(x-x_s)^2}{p_{s+1}(x,y)+}\frac{(x-x_{s+1})^2}{p_{s+2}(x,y)}+\cdots+\frac{(x-x_{n-1})^2}{p_n(x,y)}.\tag{2.8}$$

In particular, $R_0(x,y)=R(x,y)$, $R_n(x,y)=p_n(x,y)$. Then

(1) *$R_s(x_t,y_j)=\varphi_s(x_t,y_j)$ holds true for $s=0,1,\cdots,n$, $t=s,s+1,\cdots,n$, $j=0,1,\cdots,m$.*

(2)

$$\begin{aligned}\frac{\partial}{\partial x}R(x,y)=&\frac{\partial}{\partial x}p_0(x,y)+\frac{2(x-x_0)}{R_1(x,y)}-\frac{(x-x_0)^2}{R_1^2(x,y)}\Bigg\{\frac{\partial}{\partial x}p_1(x,y)\\ &+\frac{2(x-x_1)}{R_2(x,y)}-\frac{(x-x_1)^2}{R_2^2(x,y)}\Bigg(\cdots-(x-x_{n-2})^2\Bigg(\frac{\partial}{\partial x}p_{n-1}(x,y)\\ &+\frac{2(x-x_{n-1})p_n(x,y)-(x-x_{n-1})^2\frac{\partial}{\partial x}p_n(x,y)}{p_n^2(x,y)}\Bigg)\Bigg)\Bigg\}.\end{aligned}$$

For simplicity, we denote it as:

$$\begin{aligned}\frac{\partial}{\partial x}R(x,y) = {} & \frac{\partial}{\partial x}p_0(x,y) + \frac{2(x-x_0)R_1(x,y)-(x-x_0)^2\frac{\partial}{\partial x}p_1(x,y)}{R_1^2(x,y)} \\ & \frac{-}{(x-x_0)^2}\frac{2(x-x_1)R_2(x,y)-(x-x_1)^2\frac{\partial}{\partial x}p_2(x,y)}{R_2^2(x,y)} \\ & \frac{-}{(x-x_1)^2}\frac{2(x-x_2)R_3(x,y)-(x-x_2)^2\frac{\partial}{\partial x}p_3(x,y)}{R_3^2(x,y)} \\ & -\cdots \\ & \frac{-}{(x-x_{n-2})^2}\frac{2(x-x_{n-1})p_n(x,y)-(x-x_{n-1})^2\frac{\partial}{\partial x}p_n(x,y)}{p_n^2(x,y)}.\end{aligned}$$

(3)

$$\begin{aligned}\frac{\partial}{\partial y}R(x,y) = {} & \frac{\partial}{\partial y}p_0(x,y) - \frac{(x-x_0)^2}{R_1^2(x,y)} \times \left\{ \frac{\partial}{\partial y}p_1(x,y) - \frac{(x-x_1)^2}{R_2^2(x,y)} \right. \\ & \left. \times \left(\cdots - \frac{(x-x_{n-2})^2}{R_{n-1}^2(x,y)} \left(\frac{\partial}{\partial y}p_{n-1}(x,y) + \frac{-(x-x_{n-1})^2\frac{\partial}{\partial y}p_n(x,y)}{p_n^2(x,y)} \right) \right) \right\}.\end{aligned}$$

For simplicity, we denote it as:

$$\begin{aligned}\frac{\partial}{\partial y}R(x,y) = {} & \frac{\partial}{\partial y}p_0(x,y) + \frac{-(x-x_0)^2\frac{\partial}{\partial y}p_1(x,y)}{R_1^2(x,y)} \\ & \frac{+}{[-(x-x_0)^2]}\frac{-(x-x_1)^2\frac{\partial}{\partial y}p_2(x,y)}{R_2^2(x,y)} \\ & \frac{+}{[-(x-x_1)^2]}\frac{-(x-x_2)^2\frac{\partial}{\partial y}p_3(x,y)}{R_3^2(x,y)} \\ & +\cdots \\ & \frac{+}{[-(x-x_{n-3})^2]}\frac{-(x-x_{n-2})^2\frac{\partial}{\partial y}p_{n-1}(x,y)}{R_{n-1}^2(x,y)} \\ & \frac{+}{[-(x-x_{n-2})^2]}\frac{-(x-x_{n-1})^2\frac{\partial}{\partial y}p_n(x,y)}{p_n^2(x,y)}.\end{aligned}$$

Proof. (1) From (2.3) in Algorithm 2.1, Lemma 2.1 and (2.8), we get

$$\begin{aligned} R_s(x_t,y_j) &= p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{p_{s+1}(x_t,y_j)}+\cdots+\frac{(x_t-x_{t-2})^2}{p_{t-1}(x_t,y_j)}+\frac{(x_t-x_{t-1})^2}{p_t(x_t,y_j)} \\ &= p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{p_{s+1}(x_t,y_j)}+\cdots+\frac{(x_t-x_{t-2})^2}{p_{t-1}(x_t,y_j)}+\frac{(x_t-x_{t-1})^2}{\varphi_t(x_t,y_j)} \\ &= p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{p_{s+1}(x_t,y_j)}+\cdots+\frac{(x_t-x_{t-2})^2}{p_{t-1}(x_t,y_j)}+\frac{(x_t-x_{t-1})^2}{\frac{(x_t-x_{t-1})^2}{\varphi_{t-1}(x_t,y_j)-p_{t-1}(x_t,y_j)}} \\ &= p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{p_{s+1}(x_t,y_j)}+\cdots+\frac{(x_t-x_{t-2})^2}{\varphi_{t-1}(x_t,y_j)}=\cdots \\ &= p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{\varphi_{s+1}(x_t,y_j))}=p_s(x_t,y_j)+\frac{(x_t-x_s)^2}{\frac{(x_t-x_s)^2}{\varphi_s(x_t,y_j)-p_s(x_t,y_j)}} \\ &= \varphi_s(x_t,y_j). \end{aligned}$$

Moreover, (2) and (3) can be proved by induction. ■

Theorem 2.1. *Let*

$$R(x,y)=p_0(x,y)+\frac{(x-x_0)^2}{p_1(x,y)}+\frac{(x-x_1)^2}{p_2(x,y)}+\cdots+\frac{(x-x_{n-1})^2}{p_n(x,y)}, \tag{2.9}$$

where $p_i(x,y)$, $(i=0,1,\cdots,n)$ defined in Algorithm 2.1. *Then*

$$\begin{aligned} &R(x_i,y_j)=f(x_i,y_j), \quad \frac{\partial}{\partial x}R(x_i,y_j)=f_x(x_i,y_j), \\ &\frac{\partial}{\partial y}R(x_i,y_j)=f_y(x_i,y_j), \quad \forall (x_i,y_j)\in\Pi^{n,m}_{x,y}. \end{aligned}$$

Proof.

(a) From (1) of Lemma 2.2, we know

$$R(x_i,y_j)=R_0(x_i,y_j)=\varphi_0(x_i,y_j)=f(x_i,y_j), \quad \forall (x_i,y_j)\in\Pi^{n,m}_{x,y}.$$

(b) From (2) of Lemma 2.2, we get

$$\begin{aligned} \frac{\partial}{\partial x}R(x_i,y_j) =& \frac{\partial}{\partial x}p_0(x_i,y_j)+\frac{2(x_i-x_0)R_1(x_i,y_j)-(x_i-x_0)^2\frac{\partial}{\partial x}p_1(x_i,y_j)}{R_1^2(x_i,y_j)} \\ &-{}_{(x_i-x_0)^2}\frac{2(x_i-x_1)R_2(x_i,y_j)-(x_i-x_1)^2\frac{\partial}{\partial x}p_2(x_i,y_j)}{R_2^2(x_i,y_j)}-\cdots \\ &-{}_{(x_i-x_{i-3})^2}\frac{2(x_i-x_{i-2})R_{i-1}(x_i,y_j)-(x_i-x_{i-2})^2\frac{\partial}{\partial x}p_{i-1}(x_i,y_j)}{R_{i-1}^2(x_i,y_j)} \\ &-{}_{(x_i-x_{i-2})^2}\frac{2(x_i-x_{i-1})p_i(x_i,y_j)-(x_i-x_{i-1})^2\frac{\partial}{\partial x}p_i(x_i,y_j)}{p_i^2(x_i,y_j)}. \end{aligned}$$

According to (1) of Lemma 2.2 and Lemma 2.1, we derive

$$\begin{aligned}\frac{\partial}{\partial x}R(x_i,y_j)=&\frac{\partial}{\partial x}p_0(x_i,y_j)+\frac{2(x_i-x_0)\varphi_1(x_i,y_j)-(x_i-x_0)^2\frac{\partial}{\partial x}p_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)}\\&-_{(x_i-x_0)^2}\frac{2(x_i-x_1)\varphi_2(x_i,y_j)-(x_i-x_1)^2\frac{\partial}{\partial x}p_2(x_i,y_j)}{\varphi_2^2(x_i,y_j)}-\cdots\\&-_{(x_i-x_{i-3})^2}\frac{2(x_i-x_{i-2})\varphi_{i-1}(x_i,y_j)-(x_i-x_{i-2})^2\frac{\partial}{\partial x}p_{i-1}(x_i,y_j)}{\varphi_{i-1}^2(x_i,y_j)}\\&-_{(x_i-x_{i-2})^2}\frac{2(x_i-x_{i-1})\varphi_i(x_i,y_j)-(x_i-x_{i-1})^2\frac{\partial}{\partial x}\varphi_i(x_i,y_j)}{\varphi_i^2(x_i,y_j)}.\end{aligned}$$

Notice that $\varphi_i(x_i,y_j)=\frac{(x_i-x_{i-1})^2}{\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j)}$ and

$$\begin{aligned}\frac{\partial}{\partial x}\varphi_i(x_i,y_j)=&\frac{2(x_i-x_{i-1})(\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j))}{(\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j))^2}\\&-\frac{(x_i-x_{i-1})^2\left(\frac{\partial}{\partial x}\varphi_{i-1}(x_i,y_j)-\frac{\partial}{\partial x}p_{i-1}(x_i,y_j)\right)}{(\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j))^2},\end{aligned}$$

we have

$$\begin{aligned}&\frac{2(x_i-x_{i-1})\varphi_i(x_i,y_j)-(x_i-x_{i-1})^2\frac{\partial}{\partial x}\varphi_i(x_i,y_j)}{\varphi_i^2(x_i,y_j)}\\&=\frac{\partial}{\partial x}\varphi_{i-1}(x_i,y_j)-\frac{\partial}{\partial x}p_{i-1}(x_i,y_j).\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\partial}{\partial x}R(x_i,y_j)=&\frac{\partial}{\partial x}p_0(x_i,y_j)+\frac{2(x_i-x_0)\varphi_1(x_i,y_j)-(x_i-x_0)^2\frac{\partial}{\partial x}p_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)}-\cdots\\&-_{(x_i-x_{i-3})^2}\frac{2(x_i-x_{i-2})\varphi_{i-1}(x_i,y_j)-(x_i-x_{i-2})^2\frac{\partial}{\partial x}\varphi_{i-1}(x_i,y_j)}{\varphi_{i-1}^2(x_i,y_j)}.\end{aligned}$$

By repeating the above procedure, we get

$$\begin{aligned}\frac{\partial}{\partial x}R(x_i,y_j)\ =&\ \frac{\partial}{\partial x}p_0(x_i,y_j)+\frac{2(x_i-x_0)\varphi_1(x_i,y_j)-(x_i-x_0)^2\frac{\partial}{\partial x}\varphi_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)}\\=&\ \frac{\partial}{\partial x}p_0(x_i,y_j)+\frac{\partial}{\partial x}\varphi_0(x_i,y_j)-\frac{\partial}{\partial x}p_0(x_i,y_j)=\frac{\partial}{\partial x}\varphi_0(x_i,y_j).\end{aligned}$$

From Step 1 in Algorithm 2.1, we know $\varphi_0(x,y)=f(x,y)$, so

$$\frac{\partial}{\partial x}R(x_i,y_j)=f_x(x_i,y_j),\quad \forall(x_i,y_j)\in\Pi_{x,y}^{n,m}.$$

(c) From (3) of Lemma 2.2, it follows

$$\begin{aligned}\frac{\partial}{\partial y}R(x_i,y_j) = {} & \frac{\partial}{\partial y}p_0(x_i,y_j) + \frac{-(x_i-x_0)^2\frac{\partial}{\partial y}p_1(x_i,y_j)}{R_1^2(x_i,y_j)} \\ & + [-(x_i-x_0)^2]\frac{-(x_i-x_1)^2\frac{\partial}{\partial y}p_2(x_i,y_j)}{R_2^2(x_i,y_j)} + \cdots \\ & + [-(x_i-x_{i-3})^2]\frac{-(x_i-x_{i-2})^2\frac{\partial}{\partial y}p_{i-1}(x_i,y_j)}{R_{i-1}^2(x_i,y_j)} \\ & + [-(x_i-x_{i-2})^2]\frac{-(x_i-x_{i-1})^2\frac{\partial}{\partial y}p_i(x_i,y_j)}{p_i^2(x_i,y_j)}.\end{aligned}$$

Combining (1) of Lemma 2.2 and Lemma 2.1, we get

$$\begin{aligned}\frac{\partial}{\partial y}R(x_i,y_j) = {} & \frac{\partial}{\partial y}p_0(x_i,y_j) + \frac{-(x_i-x_0)^2\frac{\partial}{\partial y}p_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)} \\ & + [-(x_i-x_0)^2]\frac{-(x_i-x_1)^2\frac{\partial}{\partial y}p_2(x_i,y_j)}{\varphi_2^2(x_i,y_j)} + \cdots \\ & + [-(x_i-x_{i-3})^2]\frac{-(x_i-x_{i-2})^2\frac{\partial}{\partial y}p_{i-1}(x_i,y_j)}{\varphi_{i-1}^2(x_i,y_j)} \\ & + [-(x_i-x_{i-2})^2]\frac{-(x_i-x_{i-1})^2\frac{\partial}{\partial y}\varphi_i(x_i,y_j)}{\varphi_i^2(x_i,y_j)}.\end{aligned}$$

Notice that

$$\begin{aligned}\varphi_i(x_i,y_j) &= \frac{(x_i-x_{i-1})^2}{\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j)}, \\ \frac{\partial}{\partial y}\varphi_i(x_i,y_j) &= \frac{-(x_i-x_{i-1})^2\left(\frac{\partial}{\partial y}\varphi_{i-1}(x_i,y_j)-\frac{\partial}{\partial y}p_{i-1}(x_i,y_j)\right)}{(\varphi_{i-1}(x_i,y_j)-p_{i-1}(x_i,y_j))^2}.\end{aligned}$$

Consequently,

$$\frac{-(x_i-x_{i-1})^2\frac{\partial}{\partial y}\varphi_i(x_i,y_j)}{\varphi_i^2(x_i,y_j)} = \frac{\partial}{\partial y}\varphi_{i-1}(x_i,y_j) - \frac{\partial}{\partial y}p_{i-1}(x_i,y_j).$$

We then have

$$\begin{aligned}
\frac{\partial}{\partial y}R(x_i,y_j) &= \frac{\partial}{\partial y}p_0(x_i,y_j)+\frac{-(x_i-x_0)^2\frac{\partial}{\partial y}p_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)}\\
&\quad+[-(x_i-x_0)^2]\frac{-(x_i-x_1)^2\frac{\partial}{\partial y}p_2(x_i,y_j)}{\varphi_2^2(x_i,y_j)}\\
&\quad+\cdots+[-(x_i-x_{i-3})^2]\frac{-(x_i-x_{i-2})^2\frac{\partial}{\partial y}\varphi_{i-1}(x_i,y_j)}{\varphi_{i-1}^2(x_i,y_j)}\\
&= \cdots\\
&= \frac{\partial}{\partial y}p_0(x_i,y_j)+\frac{-(x_i-x_0)^2\frac{\partial}{\partial y}\varphi_1(x_i,y_j)}{\varphi_1^2(x_i,y_j)}\\
&= \frac{\partial}{\partial y}p_0(x_i,y_j)+\frac{\partial}{\partial y}\varphi_0(x_i,y_j)-\frac{\partial}{\partial y}p_0(x_i,y_j)\\
&= \frac{\partial}{\partial y}\varphi_0(x_i,y_j).
\end{aligned}$$

From Step 1 in Algorithm 2.1, we know $\varphi_0(x,y)=f(x,y)$. Therefore, $\frac{\partial}{\partial y}R(x_i,y_j)=f_y(x_i,y_j)$, $\forall(x_i,y_j)\in\Pi_{x,y}^{n,m}$. So the proof of Theorem 2.1 is complete. ∎

3. Characteristic properties of TWBORIs

For brevity, we cite the bivariate blending osculatory rational interpolation defined by (2.9) as TWBORIs.

Theorem 3.1. *Let $R(x,y)$ be a bivariate blending osculatory rational interpolation as in* (2.9)*, and assume that every $p_0(x,y),p_1(x,y),\cdots,p_n(x,y)$ defined in Algorithm* 2.1 *exists, with a tail-to-head rationalize evaluation for $R(x,y)$. Then polynomials $N(x,y)$ and $D(x,y)$ exist and satisfy*

$$(i).\ R(x,y)=\frac{N(x,y)}{D(x,y)},\quad D(x,y)\geq 0;\qquad (ii).\ D(x,y)|N^2(x,y).$$

Proof. Let $R_s(x,y)$ and $R(x,y)$ be defined by (2.8) and (2.9) respectively. For $s=n$,

$$R_n(x,y)=p_n(x,y)=\frac{N_n(x,y)}{D_n(x,y)},$$

which gives

$$D_n(x,y)\geq 0,\quad D_n(x,y)|N_n^2(x,y).$$

Therefore, the conclusion of Theorem 3.1 holds true for $s=n$. Now, assume

$$R_k(x,y)=\frac{N_k(x,y)}{D_k(x,y)},\quad D_k(x,y)\geq 0,\quad D_k(x,y)|N_k^2(x,y)\ \text{ valid for }\ s=k.$$

Then for $s=k-1$,

$$\begin{aligned} R_{k-1}(x,y) &= p_{k-1}(x,y)+\frac{(x-x_{k-1})^2}{R_k(x,y)}=p_{k-1}(x,y)+\frac{(x-x_{k-1})^2D_k(x,y)}{N_k(x,y)} \\ &= \frac{p_{k-1}(x,y)\frac{N_k^2(x,y)}{D_k(x,y)}+(x-x_{k-1})^2N_k(x,y)}{\frac{N_k^2(x,y)}{D_k(x,y)}}. \end{aligned}$$

Let

$$D_{k-1}(x,y)=\frac{N_k^2(x,y)}{D_k(x,y)},\quad N_{k-1}(x,y)=p_{k-1}(x,y)D_{k-1}(x,y)+(x-x_{k-1})^2N_k(x,y).$$

Thus

$$R_{k-1}(x,y)=\frac{N_{k-1}(x,y)}{D_{k-1}(x,y)},\quad D_{k-1}(x,y)\geq 0$$

and

$$\begin{aligned} N_{k-1}^2(x,y) &= p_{k-1}^2(x,y)D_{k-1}^2(x,y)+2p_{k-1}(x,y)D_{k-1}(x,y)(x-x_{k-1})^2N_k(x,y) \\ &\quad +(x-x_{k-1})^4N_k^2(x,y), \\ D_{k-1}(x,y) &= \frac{N_k^2(x,y)}{D_k(x,y)}. \end{aligned}$$

Consequently,

$$D_{k-1}(x,y)|N_{k-1}^2(x,y).$$

Choosing $k=1$, Theorem 3.1 is proved by induction. ■

Suppose $f(x,y)$ is a bivariate polynomial. We denote by $\partial_x f$ the degree of $f(x,y)$ with respect to variable x and by $\partial_y f$ the degree of $f(x,y)$ with respect to variable y [3].

Theorem 3.2. *The interpolating rational function $R(x,y)$ given in (2.9) is of degree $(n+1/n)$ with respect to x and $((n+1)(2m+1)/n(2m+1))$ with respect to y.*

Proof. It is easy to observe from (2.4)

$$\partial_x p_s(x,y)=1,\quad \partial_y p_s(x,y)=2m+1,\quad s=0,1,\cdots,n.$$

We denote by

$$R^{(s)}(x,y)=p_0(x,y)+\frac{(x-x_0)^2}{p_1(x,y)}+\frac{(x-x_1)^2}{p_2(x,y)}+\cdots+\frac{(x-x_{s-1})^2}{p_s(x,y)}=\frac{N^{(s)}(x,y)}{D^{(s)}(x,y)}.$$

For $s = 0$,

$$R^{(0)}(x,y) = p_0(x,y) = \frac{N^{(0)}(x,y)}{D^{(0)}(x,y)},$$

which gives

$$\partial_x N^{(0)}(x,y) = 1, \quad \partial_x D^{(0)}(x,y) = 0, \quad \partial_y N^{(0)}(x,y) = 2m+1, \quad \partial_y D^{(0)}(x,y) = 0.$$

For $s = 1$,

$$R^{(1)}(x,y) = p_0(x,y) + \frac{(x-x_0)^2}{p_1(x,y)} = \frac{p_0(x,y)p_1(x,y) + (x-x_0)^2}{p_1(x,y)} = \frac{N^{(1)}(x,y)}{D^{(1)}(x,y)},$$

thus

$$\partial_x N^{(1)}(x,y) = 2, \quad \partial_x D^{(1)}(x,y) = 1, \quad \partial_y N^{(1)}(x,y) = 4m+2, \quad \partial_y D^{(1)}(x,y) = 2m+1.$$

Now we assume that

$$\begin{aligned}&\partial_x N^{(s)}(x,y) = s+1, \quad \partial_x D^{(s)}(x,y) = s,\\ &\partial_y N^{(s)}(x,y) = (s+1)(2m+1), \quad \partial_y D^{(s)}(x,y) = s(2m+1).\end{aligned}$$

Then by the three-term recurrence formula, we get

$$\begin{aligned}N^{(s+1)}(x,y) &= p_{s+1}(x,y)N^{(s)}(x,y) + (x-x_s)^2 N^{(s-1)}(x,y),\\ D^{(s+1)}(x,y) &= p_{s+1}(x,y)D^{(s)}(x,y) + (x-x_s)^2 D^{(s-1)}(x,y).\end{aligned}$$

According to the hypothesis, it follows

$$\begin{aligned}\partial_x N^{(s+1)}(x,y) &= \max\{1+s+1, 2+s+1-1\} = s+2,\\ \partial_y N^{(s+1)}(x,y) &= \max\{2m+1+(s+1)(2m+1), (s+1-1)(2m+1)\}\\ &= (2m+1)(s+2),\\ \partial_x D^{(s+1)}(x,y) &= \max\{1+s+1-1, 2+s+1-2\} = s+1,\\ \partial_y D^{(s+1)}(x,y) &= \max\{2m+1+(s+1-1)(2m+1), (s+1-2)(2m+1)\}\\ &= (s+1)(2m+1).\end{aligned}$$

Therefore, the proof of Theorem 3.2 is completed by induction. ■

4. Error estimation

Let

$$R^*(x,y)=p_0^*(x,y)+\frac{(x-x_0)^2}{p_1^*(x,y)}+\cdots+\frac{(x-x_{n-1})^2}{p_n^*(x,y)}=\frac{N^*(x,y)}{D^*(x,y)}, \tag{4.1}$$

$$\begin{aligned}p_i^*(x,y)=&\varphi_i^*(x_i,y_0)+\varphi_i^*[x_i;y_0,y_0](y-y_0)+\varphi_i^*[x_i;y_0,y_0,y_1](y-y_0)^2\\&+\cdots+\varphi_i^*[x_i;y_0,y_0,\cdots,y_m,y_m](y-y_0)^2\cdots(y-y_{m-1})^2(y-y_m)\\&+\varphi_i^*[x_i;y_0,y_0,\cdots,y_m,y_m,y](y-y_0)^2\cdots(y-y_m)^2\\&+(x-x_i)(\varphi_i^*[x_i,x_i;y_0]+\varphi_i^*[x_i,x_i;y_0,y_1](y-y_0)\\&+\cdots+\varphi_i^*[x_i,x_i;y_0,\cdots,y_m](y-y_0)\cdots(y-y_{m-1})\\&+\varphi_i^*[x_i,x_i;y_0,\cdots,y_m,y](y-y_0)\cdots(y-y_m)),\end{aligned} \tag{4.2}$$

$$\varphi_i^*(x,y)=\frac{(x-x_{i-1})^2}{\varphi_{i-1}^*(x,y)-p_{i-1}^*(x,y)},\quad i=1,\cdots,n. \tag{4.3}$$

Especially, let $\varphi_0^*(x,y)=\varphi_0(x,y)=f(x,y)$.

Lemma 4.1. *Suppose every $p_i^*(x,y)$ in (4.2) exists. Then $R^*(x,y)$ defined in (4.1) satisfies:*

(1) $R^*(x_i,y)=f(x_i,y),\quad \frac{\partial}{\partial x}R^*(x_i,y)=f_x(x_i,y),\quad i=0,1,\cdots,n.$

(2) $R^*(x,y_j)=R(x,y_j),\quad j=0,1,\cdots,m.$

Proof.

(1). (i) From (4.2), we have

$$\begin{aligned}p_i^*(x_i,y)=&\varphi_i^*(x_i,y_0)+\varphi_i^*[x_i;y_0,y_0](y-y_0)+\varphi_i^*[x_i;y_0,y_0,y_1](y-y_0)^2\\&+\cdots+\varphi_i^*[x_i;y_0,y_0,\cdots,y_m,y_m](y-y_0)^2\cdots(y-y_{m-1})^2(y-y_m)\\&+\varphi_i^*[x_i;y_0,y_0,\cdots,y_m,y_m,y](y-y_0)^2\cdots(y-y_m)^2.\end{aligned}$$

By Definition 2.1, it follows

$$p_i^*(x_i,y)=\varphi_i^*(x_i,y),\quad i=0,1,\cdots,n. \tag{4.4}$$

Let

$$R_s^*(x,y)=p_s^*(x,y)+\frac{(x-x_s)^2}{p_{s+1}^*(x,y)}+\frac{(x-x_{s+1})^2}{p_{s+2}^*(x,y)}+\cdots+\frac{(x-x_{n-1})^2}{p_n^*(x,y)},$$

for $s=0,1,\cdots,n$, $t=s,s+1,\cdots,n$. According to (4.3) and (4.4), we derive

$$\begin{aligned} R_s^*(x_t,y) &= p_s^*(x_t,y)+\frac{(x_t-x_s)^2}{p_{s+1}^*(x_t,y)}+\cdots+\frac{(x_t-x_{t-1})^2}{p_t^*(x_t,y)} \\ &= p_s^*(x_t,y)+\frac{(x_t-x_s)^2}{p_{s+1}^*(x_t,y)}+\cdots+\frac{(x_t-x_{t-1})^2}{\varphi_t^*(x_t,y)} \\ &= p_s^*(x_t,y)+\frac{(x_t-x_s)^2}{p_{s+1}^*(x_t,y)}+\cdots+\frac{(x_t-x_{t-1})^2}{\frac{(x_t-x_{t-1})^2}{\varphi_{t-1}^*(x_t,y)-p_{t-1}^*(x_t,y)}} \\ &= \cdots=\varphi_s^*(x_t,y). \end{aligned}$$

Consequently,

$$R_s^*(x_t,y)=\varphi_s^*(x_t,y),\quad s=0,1,\cdots,n,\quad t=s,s+1,\cdots,n. \tag{4.5}$$

Choosing $s=0$, we get $R^*(x_i,y)=\varphi_0^*(x_i,y)=f(x_i,y)$.
(ii) From (2) of Lemma 2.2, it follows

$$\begin{aligned} \frac{\partial}{\partial x}R^*(x_i,y)=&\frac{\partial}{\partial x}p_0^*(x_i,y)+\frac{2(x_i-x_0)R_1^*(x_i,y)-(x_i-x_0)^2\frac{\partial}{\partial x}p_1^*(x_i,y)}{R_1^{*2}(x_i,y)} \\ &-(x_i-x_0)^2\frac{2(x_i-x_1)R_2^*(x_i,y)-(x_i-x_1)^2\frac{\partial}{\partial x}p_2^*(x_i,y)}{R_2^{*2}(x_i,y)}-\cdots \\ &-(x_i-x_{i-2})^2\frac{2(x_i-x_{i-1})p_i^*(x_i,y)-(x_i-x_{i-1})^2\frac{\partial}{\partial x}p_i^*(x_i,y)}{p_i^{*2}(x_i,y)}. \end{aligned} \tag{4.6}$$

We get from (4.2),

$$\frac{\partial}{\partial x}p_i^*(x_i,y)=\varphi_i^*[x_i,x_i;y]=\frac{\partial}{\partial x}\varphi_i^*(x_i,y). \tag{4.7}$$

Combining (4.3)-(4.7) and using the similar method used in the proof of (b) in Theorem 2.1, we conclude that

$$\frac{\partial}{\partial x}R^*(x_i,y)=\frac{\partial}{\partial x}\varphi_0^*(x_i,y)=f_x(x_i,y)$$

holds for $i=0,1,\cdots,n$.

(2). It can be verified that

$$R(x,y_j)=p_0(x,y_j)+\frac{(x-x_0)^2}{p_1(x,y_j)+}\frac{(x-x_1)^2}{p_2(x,y_j)+}\cdots+\frac{(x-x_{n-1})^2}{p_n(x,y_j)}, \tag{4.8}$$

$$R^*(x,y_j)=p_0^*(x,y_j)+\frac{(x-x_0)^2}{p_1^*(x,y_j)+}\frac{(x-x_1)^2}{p_2^*(x,y_j)+}\cdots+\frac{(x-x_{n-1})^2}{p_n^*(x,y_j)}, \tag{4.9}$$

$$\begin{aligned} p_i(x,y_j)&=\varphi_i(x_i,y_j)+(x-x_i)\varphi_i[x_i,x_i;y_j]\\ &=\varphi_i(x_i,y_j)+(x-x_i)\frac{\partial}{\partial x}\varphi_i(x_i,y_j), \end{aligned} \tag{4.10}$$

$$\begin{aligned} p_i^*(x,y_j)&=\varphi_i^*(x_i,y_j)+(x-x_i)\varphi_i^*[x_i,x_i;y_j]\\ &=\varphi_i^*(x_i,y_j)+(x-x_i)\frac{\partial}{\partial x}\varphi_i^*(x_i,y_j). \end{aligned} \tag{4.11}$$

For $\varphi_0^*(x,y)=\varphi_0(x,y)$, $p_0(x,y_j)=p_0^*(x,y_j)$. For

$$\varphi_1^*(x,y_j)=\frac{(x-x_0)^2}{\varphi_0^*(x,y_j)-p_0^*(x,y_j)}=\frac{(x-x_0)^2}{\varphi_0(x,y_j)-p_0(x,y_j)}=\varphi_1(x,y_j),$$

so

$$\frac{\partial}{\partial x}\varphi_1^*(x,y_j)=\frac{\partial}{\partial x}\varphi_1(x,y_j).$$

According to (4.10) and (4.11), we have $p_1(x,y_j)=p_1^*(x,y_j)$. Similarly, we get

$$p_i(x,y_j)=p_i^*(x,y_j),\quad i=0,1,\cdots,n,\quad j=0,1,\cdots,m,$$

by (4.8) and (4.9). Thus $R(x,y_j)=R^*(x,y_j)$ holds for $j=0,1,\cdots,m$.

Theorem 4.1. *(Error estimation) Suppose* $D=[a,b]\times[c,d]$, $X_n=\{x_0,x_1,\cdots,x_n\}\subset[a,b]\subset R$, $Y_n=\{y_0,y_1,\cdots,y_m\}\subset[c,d]\subset R$, $\Pi_{x,y}^{n,m}=X_n\times Y_m$, *and* $f(x,y)\in C^{2n+m+3}(D)$. *Let* $R(x,y)$,$R^*(x,y)$, $N(x,y)$,$D(x,y)$,$N^*(x,y)$ *and* $D^*(x,y)$ *be defined in Theorem* 3.1 *and* (4.1) *respectively. Let*

$$\begin{aligned} &E(x,y)=D(x,y)D^*(x,y)(f(x,y)-R(x,y)),\\ &u_n(x)=(x-x_0)(x-x_1)\cdots(x-x_n),\\ &v_m(y)=(y-y_0)(y-y_1)\cdots(y-y_m). \end{aligned}$$

Then

$$\begin{aligned} E(x,y)=&\frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E(\xi,y)}{\partial x^{2n+2}}+\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}E(x,\eta)}{\partial y^{m+1}}\\ &-\frac{u_n^2(x)}{(2n+2)!}\frac{v_m(y)}{(m+1)!}\frac{\partial^{2n+m+3}E(\xi,\eta)}{\partial x^{2n+2}\partial y^{m+1}}, \end{aligned}$$

where $\xi\in(a,b)$ *and* $\eta\in(c,d)$.

Proof. Let

$$E_1(x,y)=D(x,y)D^*(x,y)(f(x,y)-R^*(x,y)),$$
$$E_2(x,y)=D(x,y)D^*(x,y)(R^*(x,y)-R(x,y)).$$

By (1) of Lemma 4.1, we have $E_1(x_i,y)=0$, $\frac{\partial}{\partial x}E_1(x_i,y)=0$. Consequently,

$$E_1(x,y)=\frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E_1(\xi,y)}{\partial x^{2n+2}}, \tag{4.12}$$

where $u_n(x)=(x-x_0)(x-x_1)\cdots(x-x_n)$, ξ is located among $x_0,x_1,\cdots,x_n,x$.

From (2) of Lemma 4.1, it follows $E_2(x,y_j)=0$, $j=0,1,\cdots,m$. So

$$E_2(x,y)=\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}E_2(x,\eta)}{\partial y^{m+1}}, \tag{4.13}$$

where $v_m(y)=(y-y_0)(y-y_1)\cdots(y-y_m)$, η is located among $y_0,y_1,\cdots,y_m,y$. Notice that

$$\begin{aligned}E_2(x,y) &= D(x,y)D^*(x,y)(R^*(x,y)-R(x,y))\\ &= D(x,y)N^*(x,y)-D^*(x,y)N(x,y).\end{aligned}$$

Hence,

$$\partial_x E_2(x,y)=\max\{2n+1,2n+1\}=2n+1,$$

which gives

$$\frac{\partial^{2n+2}E(x,y)}{\partial x^{2n+2}}=\frac{\partial^{2n+2}E_1(x,y)}{\partial x^{2n+2}}. \tag{4.14}$$

Combining (4.12)-(4.14), we derive

$$\begin{aligned}E(x,y) &= E_1(x,y)+E_2(x,y)\\
&= \frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E_1(\xi,y)}{\partial x^{2n+2}}+\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}E_2(x,\eta)}{\partial y^{m+1}}\\
&= \frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E(\xi,y)}{\partial x^{2n+2}}+\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}(E(x,\eta)-E_1(x,\eta))}{\partial y^{m+1}}\\
&= \frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E(\xi,y)}{\partial x^{2n+2}}+\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}E(x,\eta)}{\partial y^{m+1}}\\
&\quad -\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}}{\partial y^{m+1}}\left[\frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E_1(\xi,\eta)}{\partial x^{2n+2}}\right]\\
&= \frac{u_n^2(x)}{(2n+2)!}\frac{\partial^{2n+2}E(\xi,y)}{\partial x^{2n+2}}+\frac{v_m(y)}{(m+1)!}\frac{\partial^{m+1}E(x,\eta)}{\partial y^{m+1}}\\
&\quad -\frac{u_n^2(x)}{(2n+2)!}\frac{v_m(y)}{(m+1)!}\frac{\partial^{2n+m+3}E(\xi,\eta)}{\partial x^{2n+2}\partial y^{m+1}}. \qquad \blacksquare\end{aligned}$$

Table 5.1: Data sets.

	$x_0=2$	$x_0=2$	$x_1=3$	$x_1=3$
$y_0=2$	$f(x_0,y_0)=1.5$	$f_x(x_0,y_0)=0.9375$	$f(x_1,y_0)=2.8182$	$f_x(x_1,y_0)=1.686$
$y_0=2$	$f_y(x_0,y_0)=0.3125$		$f_y(x_1,y_0)=0.1074$	
$y_1=3$	$f(x_0,y_1)=1.8889$	$f_x(x_0,y_1)=0.7037$	$f(x_1,y_1)=3$	$f_x(x_1,y_1)=1.5$
$y_1=3$	$f_y(x_0,y_1)=0.4568$		$f_y(x_1,y_1)=0.25$	

5. Numerical example

Example 5.1. Suppose $(x_i,y_j), f(x_i,y_j), f_x(x_i,y_j)$ and $f_y(x_i,y_j)$ are given in the Table 5.1. Let $\varphi_0(x,y)=f(x,y)$. We want to find a polynomial $p_0(x,y)$, such that

	$x_0=2$	$x_0=2$
$y_0=2$	$p_0(2,2)=\varphi_0(2,2)=1.5$	$\frac{\partial}{\partial x}p_0(2,2)=\frac{\partial}{\partial x}\varphi_0(2,2)=0.9375$
$y_0=2$	$\frac{\partial}{\partial y}p_0(2,2)=\frac{\partial}{\partial y}\varphi_0(2,2)=0.3125$	
$y_1=3$	$p_0(2,3)=\varphi_0(2,3)=1.8889$	$\frac{\partial}{\partial x}p_0(2,3)=\frac{\partial}{\partial x}\varphi_0(2,3)=0.7037$
$y_1=3$	$\frac{\partial}{\partial y}p_0(2,3)=\frac{\partial}{\partial y}\varphi_0(2,3)=0.4568$	

Using Step 1 in Algorithm 2.1 gives

$$p_0(x,y)=\frac{7}{8}+\frac{5}{16}y+\frac{191}{2500}(y-2)^2+\left(-\frac{19140298416325}{2251799813685248}y+\frac{57420895248975}{2251799813685248}\right)(y-2)^2+(x-2)\left(\frac{14051}{10000}-\frac{1169}{5000}y\right).$$

Let $\varphi_1(x,y)=\frac{(x-2)^2}{\varphi_0(x,y)-p_0(x,y)}$. We wish to find a polynomial $p_1(x,y)$, such that

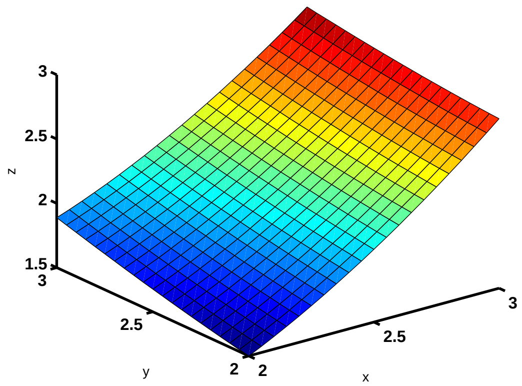
	$x_1=3$	$x_1=3$
$y_0=2$	$p_1(3,2)=\varphi_1(3,2)=2.6269$	$\frac{\partial}{\partial x}p_1(3,2)=\frac{\partial}{\partial x}\varphi_1(3,2)=0.0891$
$y_0=2$	$\frac{\partial}{\partial y}p_1(3,2)=\frac{\partial}{\partial y}\varphi_1(3,2)=-0.1983$	
$y_1=3$	$p_1(3,3)=\varphi_1(3,3)=2.4546$	$\frac{\partial}{\partial x}p_1(3,3)=\frac{\partial}{\partial x}\varphi_1(3,3)=0.1115$
$y_1=3$	$\frac{\partial}{\partial y}p_1(3,3)=\frac{\partial}{\partial y}\varphi_1(3,3)=-0.1627$	

Using Step 2 in Algorithm 2.1 gives

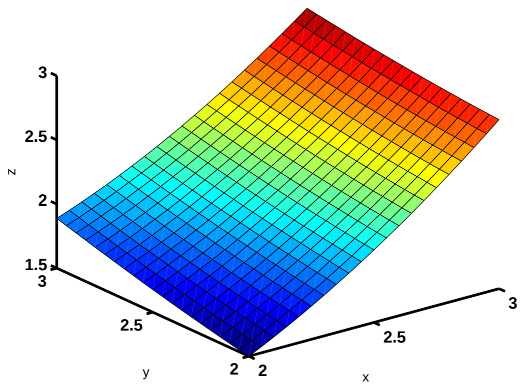
$$p_1(x,y)=\frac{6047}{2000}-\frac{1983}{10000}y+\frac{936748722493067}{36028797018963968}(y-2)^2+\left(-\frac{590872271111019}{36028797018963968}y+\frac{1772616813333057}{36028797018963968}\right)(y-2)^2+(x-3)\left(\frac{443}{10000}+\frac{14}{625}y\right).$$

Therefore $R(x,y)=p_0(x,y)+\frac{(x-2)^2}{p_1(x,y)}$.

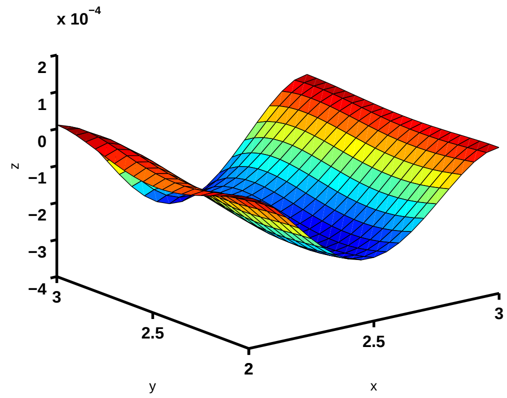
In Fig. 5.1, the interpolation function $R(x,y)$ and the corresponding error function are plotted. It is seen that the error is small, which implies that the proposed algorithm is efficient.



(a)



(b)



(c)

Fig. 5.1. (a) $f(x,y)=\frac{x^3+y^2}{3x+y}$. (b) Interpolating function $R(x,y)$. (c) Error function.

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