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Stability and Convergence of an Implicit Difference Approximation for the Space Riesz Fractional Reaction-Dispersion Equation

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Abstract

In this paper, we consider a Riesz space-fractional reaction-dispersion equation (RSFRDE). The RSFRDE is obtained from the classical reaction-dispersion equation by replacing the second-order space derivative with a Riesz derivative of order $\beta \in (1,2]$. We propose an implicit finite difference approximation for RSFRDE. The stability and convergence of the finite difference approximations are analyzed. Numerical results are found in good agreement with the theoretical analysis.

Keywords: Riesz fractional derivative; fractional reaction-dispersion equation; implicit finite difference approximation; stability; convergence.
Mathematics subject classification: 26A33, 35K57, 65M12

1. Introduction

Recently, a growing number of works have been concerned with dynamical systems described by fractional-order equations which involve derivatives and integrals of non-integer order. It is found and testified that many phenomena in engineering, physics, finance, hydrology, chemistry and other sciences [1,2] can be simulated by fractional-order equations. These models are more adequate than the previously used integer-order models. Fractional order derivatives and integrals provide a powerful instrument for the description of memory and hereditary properties of difference substances. Some partial differential equations of space-time fractional order were successfully used for modeling relevant physical process and financial behavior. For example, it can be used in groundwater hydrology to model the transport of passive traces carried by fluid flow in a porous medium [3, 4]. Furthermore, it can be used in financial markets to model the high-frequency price dynamics [5, 6]. We can also interpret the two-level difference scheme resulting from the Grunwald-Letniko discretization of fractional derivatives as a random walk model discrete in space and time [7].

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The basic analytic theory for the space-fractional diffusion processes was developed in 1952 by Fell [8] via inversion of Riesz potential operators. Mainardi et al. [9] considered the space fractional diffusion equation and presented an explicit representation of the Green function for the equation. Benson et al. [10, 11] considered the space fractional advection-dispersion equation and gave an analytic solution in terms of the α-stable error function. Liu et al. [12] derived the complete solution of the time fractional advection-dispersion equation. However, numerical methods and theoretical analysis of fractional differential equations are still at an early stage of development. Lynch et al. [13] presented two different discretization methods for the fractional diffusion equation, but stability and convergence are not presented. Lin and Liu [14] proposed higher-order approximations of a nonlinear fractional-order ordinary differential equation with initial value and proved consistency, convergence and stability of the fractional higher-order methods. Shen and Liu [15] considered the space-fractional diffusion equation and gave error analysis. Liu et al. [16] presented the numerical solution of a space fractional Fokker-Planck equation. Meerschaert et al. [17] considered the finite difference approximations for two-sided space-fractional partial differential equations and discussed their stability, consistency and convergence of the method. It is also noticed that fractional reaction-diffusion equations can be used to model activator-inhibitor dynamics with anomalous diffusion, which occurs in spatially inhomogeneous media [18].

In this paper we consider a Riesz space-fractional reaction-dispersion equation in a bounded space domain and time domain. We present an implicit difference approximation for the equation and analyze its stability and convergence. Furthermore, to evaluate the efficiency of the obtained difference scheme, a comparison with method of lines (MoL) is provided. The MoL was first introduced to solve space fractional partial differential equations by Liu et al. [16, 19]. Finally, some numerical examples are given to show that the numerical results are in good agreement with our theoretical analysis.

2. An explicit finite approximation for RSFRDE

In this section, we consider RSFRDE in a bounded space domain $[0, L]$ with the following initial and boundary conditions:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = -u(x,t) + {}_x D_0^{\beta} u(x,t), & 0 < x < L,\ 0 \le t \le T, \\ u(x,0) = g(x), & 0 \le x \le L, \\ \dfrac{\partial u(0,t)}{\partial x} = 0, \quad u(L,t) = 0, & 0 \le t \le T, \end{cases} \tag{2.1}$$

where $1 < \beta \le 2$, and we assume that both $u(x,t)$ and $g(x)$ are real-valued and sufficiently well-behaved functions, the Riesz space-fractional derivative of order β, ${}_x D_0^{\beta}$, is defined by analytic continuation in the whole range $1 < \beta \le 2$ [7]

$${}_x D_0^{\beta} := -I^{-\beta} = -c(I_+^{-\beta} + I_-^{-\beta}), \tag{2.2}$$