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THE WELL POSEDNESS IN A PARABOLIC MULTIPLE FREE BOUNDARY PROBLEM*

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Abstract We consider a free boundary problem in a parabolic partial differential equation with multiple interfacial curves which is reduced to a reaction-diffusion equation. The forcing term of this problem is not continuously differentiable and thus we use Green's function to make a regular one. The existence, uniqueness and dependence on initial conditions will be shown in this paper.
Key Words Evolution equation; free boundary problem; parabolic equation.
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1. Introduction

The reaction diffusion systems with a small layer parameter ε and a controlling parameter τ are described by

$$\begin{aligned} \varepsilon\tau u_t &= \varepsilon^2 u_{xx} + f(u,v) \\ v_t &= Dv_{xx} + g(u,v), \quad (x,t) \in (0,1) \times (0,\infty) \end{aligned} \tag{1}$$

Here u and v measure the levels of two diffusing quantities. The functions u and v satisfy Neumann boundary conditions at $x = 0, 1$. The reaction terms are assumed to be of the bistable type which means that the nullcline of f and g has three intersection points and the curve $f = 0$ determines a triple valued function of v. This system is a model of the time evolution of interaction between two separated populations and also a model of the mixing of chemically reacting-diffusing substances.

When ε and τ are chosen to be very small, the system (1) models a situation in which the quantity measured by u reacts much faster than that measured by v (τ small), while at the same time u diffuses slower than v (ε small). The principal interest in systems like (1) comes from the fact that there exist families of stationary solutions parametrized by ε, which approach discontinuous functions of x as $\varepsilon \to 0$. When ε

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is small, the stationary solution, being smooth, exhibits an abrupt but continuously differentiable transition at the location of the limiting discontinuity. The transition takes place in an x-interval of length $O(\varepsilon)$. An x-interval, in which such an abrupt change takes place, is loosely called a layer — a boundary layer when it is adjacent to an endpoint of the interval or an internal layer when it is in the interior of the interval.

Whenever the singular limit $\varepsilon \downarrow 0$ of the system (1), an analysis of the layer solutions suggests that the layer of width $O(\varepsilon)$ converges to interfacial curves $x = s^1(t), \cdots, x = s^n(t)$ in x,t-space as $\varepsilon \downarrow 0$. In [1], the authors have shown the existence and uniqueness for a parabolic partial differential equation with a single free boundary, when the kinetics f and g are of Mckean type:

$$f(u,v) = -u + H(u-a) - v \quad \text{and } g(u,v) = kv - u$$

When $\varepsilon = 0$, the problem (1) with applying these dynamics of f and g is reduced to a free boundary problem with multiple interfacial curves which were introduced in [2].

$$\begin{cases} v_t = Dv_{xx} - c^2 v + \sum_{i=1}^{n} (-1)^{n+1} H(x - s^i(t)) & \text{for } (x,t) \in \Omega^- \cup \Omega^+ \\ v_x(0,t) = 0 = v_x(1,t) & \text{for } t > 0 \\ v(x,0) = v_0(x) & \text{for } 0 \le x \le 1 \\ \tau \dfrac{ds^i}{dt} = (-1)^{i+1} C(v(s^i(t),t)) & \text{for } t > 0,\ i = 1,2,\cdots,n \\ s^i(0) = s_0^i, & i = 1,2,\cdots,n \end{cases} \tag{2}$$

where $v(x,t)$ and $v_x(x,t)$ are assumed to be continuous in Ω. Here $H(y)$ is the Heaviside function, $\Omega = (0,1) \times (0,\infty)$, $\Omega^- = \{(x,t) \in \Omega : x \in \bigcup_{i=0,2,4,\cdots} (s^i(t), s^{i+1}(t))\}$ and $\Omega^+ = \{(x,t) \in \Omega : x \in \bigcup_{i=1,3,5,\cdots} (s^i(t), s^{i+1}(t))\}$ where $s^0(t) = 0$ and $s^n(t) = 1$.

In this paper, we show a well-posedness in a free boundary problem for a parabolic partial differential equation of (1). The approach to the problem of the well posedness is to apply the semigroup theory of nonlinear evolution equation. We now write (2) as an abstract evolution equation

$$\frac{d(v, s^1, s^2, \cdots, s^n)}{dt} + A(v, s^1, s^2, \cdots, s^n) = F(v, s^1, s^2, \cdots, s^n)$$
$$(v, s^1, s^2, \cdots, s^n)(0) = (v_0(\cdot), s_0^1, s_0^2, \cdots, s_0^n)$$

These are a differential equation in a space $\widetilde{X}$ of the form $\widetilde{X} = X \times J^n$, where X is a space of functions and J is a real interval. For the problem (2) this could be done by