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THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTIONS OF A DEGENERATE REACTION-DIFFUSION PROBLEM*

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Abstract Consider in this paper the existence, uniqueness and asymptotic behaviour of the solutions of the mixed problem for a class of degenerate quasilinear parabolic equations and properties of the corresponding stationary solutions.

Key Words Variational principle; stationary solution; equilibrium solution; asymptotic behaviour, region attractivity.

Classification 35K60, 35K65

1. Introduction

Let $D \subset R^n$ be a bounded domain with the boundary $\partial D \in C^1$. Consider the non-negative solutions of the problem

$$\begin{cases} w_t - L\Phi(w) = a(x)f(w), & (x,t) \in D \times (0,t) \\ w = \chi, & (x,t) \in \partial D \times [0,T) \\ w = w_0, & (x,t) \in \overline{D} \times \{0\} \end{cases} \tag{P}$$

where $L = \partial_i(a^{ij}\partial_j)$ with $a^{ij} = a^{ij}(x) = a^{ji}(x) \in C(\overline{D})$ is a uniformly elliptic operator, i.e., there are constants Λ, λ such that $\Lambda \geq \lambda > 0$, and for any $\xi \in R^n$ it holds that $\Lambda|\xi|^2 \geq a^{ij}\xi_i\xi_j \geq \lambda|\xi|^2$. Here and throughout the notation ∂_i is used for $\dfrac{\partial}{\partial x_i}$, and the summation convention over twice repeated indices is often used. Suppose that

(H1) $\Phi \in C^1[0,\infty)$ is a monotonically increasing function satisfying that $\Phi(0) = \Phi'(0) = 0$ and Φ^{-1} is Hölder continuous;

(H2) $f \in C^1[0,\infty)$ is an increasing function with $f(0) = 0$;

(H3) $a \in C^0(\overline{D}), w_0 \in L^\infty(D), \chi \in C^1(\partial D \times (0,T))$ and $\chi = w_0$ on $\partial D \times \{0\}$.

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Problems of this type arise in population dynamics and in reaction-diffusion processes. The investigation of this paper was motivated by the special case of that (see [1] and [2])

$$u_t = \Delta\Phi(u) + f(\lambda, u), \lambda \in R; u_t = \Delta\Phi(u) = uf(x,t), (x,t) \in D \times (0,\infty)$$

On the former equation, the authors have proved a uniqueness and existence theorem for its mixed problem. Moreover, the asymptotic behaviour of the solution when $t \to \infty$ has been discussed for some values of λ. For the mixed problem of the later equation, the authors have given a good discription for the solution and the corresponding stationary solution. In this paper, based on above two works, we consider, for more general degenerate quasilinear parabolic equations, the properties of the solution of the problem (P) as well as the corresponding stationary solution.

Definition 1.1 *We call $w \in L^\infty(D)$ is an equilibrium solution of problem (P) if for any $\eta \in C^2(\overline{D})$ with zero value on ∂D, it holds that*

$$-\int_D \Phi(w) L\eta dx + \int_{\partial D} \Phi(\chi)\frac{\partial \eta}{\partial \nu} ds = \int_D a(x) f(w)\eta(x) dx \tag{1.1}$$

where $\frac{\partial \eta}{\partial \nu} = a^{ij}\frac{\partial \eta}{\partial x_i}\cos(n, x_j)$ is the oblique derivative of η. By an upper (lower) solution $\overline{w} \in L^\infty(D)(\underline{w} \in L^\infty(D))$ of (1.1) (i.e., by an equilibrium upper (lower) solution of (P)) we mean that $\overline{w}(\underline{w})$ satisfies (1.1) with the inequality sign $\geq (\leq)$, for any positive η as above.

Because of Hölder continuity of Φ^{-1}, we know from [2] that any solution of (1.1) is a classical solution, and $u = \Phi(w) \in C^{2,\alpha}(\overline{D})$ is a classical solution of the problem

$$\begin{cases} Lu + a(x)g(u) = 0, & in\ D \\ u = \varphi, & on\ \partial D \end{cases} \tag{P_s}$$

where $g = f \circ \Phi^{-1}, \varphi = \Phi[\chi(\cdot)]$. We call such u as a *stationary solution* of problem (P). u is called as an *upper (lower) solution* of (P_s) if it satisfies

$$Lu + a(x)g(u) \leq (\geq) 0 \ in\ D, u \geq (\geq)\varphi \ on\ \partial D$$

Definition 1.2 *$w \in C([0,T]; L^1(D)) \cap L^\infty(Q_T)$ is called as a solution of (P) provided that for any $t \in [0,T]$ and non-negative $\zeta \in C^2(\overline{Q}_t)$ with $\zeta = 0$ on $\partial D \times (0,T)$ it holds that*

$$\int_D w(x,t)\zeta(x,t)dx - \int_{Q_t}[w\zeta_t + \Phi(w)L\zeta]dxdt + \int_0^t dt\int_{\partial D}\Phi(\chi)\frac{\partial\zeta}{\partial\nu}ds$$
$$= \int_D w_0\zeta(x,0)dx + \int_{Q_t} a(x)f(w)\zeta(x,t)dxdt \tag{1.2}$$

where $Q_t = D\times(0,t)$ and $\frac{\partial}{\partial\nu}$ is as the one as in Definition 1.1. Instead of sign "=" in (1.2) by "$\geq (\leq)$", then w is called as an upper (lower) solution of (P).