J. Partial Diff. Eqs., Vol.6, No.3(1993), 339–348

THE HYPOELLIPTICITY AND PARAMETRICES FOR A CLASS OF NONPRINCIPAL TYPE LPDOs

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(Received June 11, 1991)

Abstract In this paper, we employ the method of increment operators, by means of the left invariant operators $\mathcal{L}^{\nu,s}$ on nilpotent Lie group $G^{d_1+d_2}$ and the calculus of $\nu - \Psi DOs$, to discuss the hypoellipticity for a class of nonprincipal type LPDOs $\mathcal{L}$ on a $d_1 + d_2$ dimensional manifold M. For those hypoelliptic $\mathcal{L}$ we also construct their parametrices.
Key Words Increment operators; LPDOs; hypoellipticity; parametrices
Classification 35G.

0. Introduction

In the classical theory of LPDOs, the study for hypoellipticity and parametrices of operators with variable coefficients occupies a fairly important position. It is well-known that this is a knotty problem so that less results have been obtained. But in recent years, with the rise of harmonic analysis on nilpotent Lie groups, the study on such subjects for some operators constructed from vector fields has made considerable headway.

This paper is devoted to hypoellipticity and parametrices for a class of nonprincipal type LPDOs $\mathcal{L}$ on a d_1+d_2 dimensional manifold M. In local coordinates, the operator $\mathcal{L}$ has the form

$$\mathcal{L} = -\sum_{j=1}^{d_1} X_j^2 - iV + C \tag{0.1}$$

where the $X_j, j = 1, 2, \cdots, d_1$, are real C^∞ vector fields and pointwise linearly independent, V is a complex C^∞ vector field, and C is a complex valued C^∞ function. The operator $\mathcal{L}$ has more general form

$$\mathcal{L} = -\sum_{j,k=1}^{d_1} g_{jk} X_j X_k - iV + C$$

where $[g_{ik}]$ is a real C^∞ function matrix of pointwise positive definite form. This class of operators includes elliptic and parabolic operators and the Schrödinger operators on the Euclidean group as well as the Kohn-Laplacian and heat operators and the Schrödinger operators on the Heisenberg group. By means of the method of increment operators (proposed first by Professor Luo Xuebo in [4]), we first discuss the invertibility of the left invariant operator $\mathcal{L}_\lambda^{y,s}$ (as a model operator of $\mathcal{L}$ at the point (y,s)) on the nilpotent Lie group $G^{d_1+d_2}$. Then, the hypoellipticity of $\mathcal{L}$ follows from the inverse of $\mathcal{L}_\lambda^{y,s}$ and calculus of $\nu-\Psi DOs$. Besides, a parametrix of $\mathcal{L}$ is constructed when $\mathcal{L}$ is hypoelliptic.

The main idea of this paper comes from Beals and Greiner's one in [1] where the operator $\mathcal{L}$ of (0.1) with the case $d_2=1$ was discussed. A key of their work is to convert a skew-symmetric matrix into one with normal form by a unitary change of coordinates. But this cannot be carried out for the case $d_2\geq 2$, since any unitary transform of coordinates, in general, cannot simultaneously convert d_2 skew-symmetric matrices into ones in normal form. By means of the result of [1], we introduce a family of increment operators to obtain the desirable results. Thus, our work is a nontrivial extension for the case $d_2=1$.

1. The Model Operator $\mathcal{L}^{y,s}$ of the Operator $\mathcal{L}$

Let M be a d_1+d_2 dimensional C^∞ manifold, and let $\mathcal{L}$ have the form (0.1). Locally we can choose vector fields $T_l, l=1,2,\cdots,d_2$, so that span $\{X_1,\cdots,X_{d_1},T_1,\cdots,T_{d_2}\}=TM$. Then (0.1) may be rewritten as

$$\mathcal{L}=-\sum_{j=1}^{d_1}X_j^2-i\sum_{l=1}^{d_2}\lambda_l T_l+\sum_{j=1}^{d_1}\gamma_j X_j+C \tag{1.1}$$

where λ_j,γ_j and C are complex-valued C^∞ functions.

Let U be a coordinate neighborhood, for given $(y,s)\in U$ we can choose unique affine coordinates $(x,t)=(x_1,\cdots,x_{d_1},t_1,\cdots,t_{d_2})$ such that $(x(y,s),t(y,s))=(0,0)$ and

$$\begin{cases} X_j=\dfrac{\partial}{\partial x_j}+\dfrac{1}{2}\Big[\displaystyle\sum_{k=1}^{d_1}\alpha_{jk}(x,t)\frac{\partial}{\partial x_k}+\sum_{k=1}^{d_2}\alpha_{j,d_1+k}(x,t)\frac{\partial}{\partial t_k}\Big], & j=1,2,\cdots,d_1\\ T_l=\dfrac{\partial}{\partial t_l}+\dfrac{1}{2}\Big[\displaystyle\sum_{k=1}^{d_1}\alpha_{d_1+l,k}(x,t)\frac{\partial}{\partial x_k}+\sum_{k=1}^{d_2}\alpha_{d_1+l,d_1+k}(x,t)\frac{\partial}{\partial t_k}\Big], & l=1,2,\cdots,d_2\end{cases}$$

where $\alpha_{jk}(y,s)=(0,0), j,k=1,2,\cdots,d_1+d_2,(x,t)\in U$. These coordinates are called the (y,s)-coordinates.

Put

$$a_{jk}^{(l)}=a_{jk}^{(l)}(y,s)=\frac{\partial}{\partial x_k}\alpha_{j,d_1+l}(y,s) \tag{1.2}$$

and

$$A_l=[a_{jk}^{(l)}]$$

$1\leq j,k\leq d_1, 1\leq l\leq d_2$. We define a binary composition on $R^{d_1+d_2}$ by

$$(x^{(1)},t^{(1)})\cdot(x^{(2)},t^{(2)})=(x^{(1)}+x^{(2)},t^{(1)}+t^{(2)}+\frac{1}{2}x^{(2)}Ax^{(1)}) \tag{1.3}$$