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BOUNDARY VALUE PROBLEMS FOR NONLINEAR ELLIPTIC EQUATIONS IN CYLINDRICAL DOMAINS

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In this paper we consider weak solutions of the equation

$$L(u) \equiv \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\Big(a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i} = |u|^{p-1}u \tag{1}$$

in a domain

$$S(0,\infty) = \{x : \hat{x} \in \omega, 0 < x_n < \infty\}$$

with the boundary condition

$$\frac{\partial u}{\partial \gamma} \equiv \sum_{i,j=1}^{n} a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\nu_i = 0 \quad \text{on} \quad \sigma(0,\infty) \tag{2}$$

where $x = (x_1, \cdots, x_n)$, $\hat{x} = (x_1, \cdots, x_{n-1})$, ω is a bounded domain in $R_{\hat{x}}^{n-1}$ with a smooth boundary $\partial\omega$,

$\sigma(0,\infty) = \{x : \hat{x} \in \partial\omega,\ 0 < x_n < \infty\}$, $\nu = (\nu_1, \cdots, \nu_n)$ is the exterior unit normal to $\sigma(0,\infty)$, $a_{ij}(\hat{x})$, $a_i(\hat{x})$ are measurable bounded functions,

$$m|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(\hat{x})\xi_i\xi_j \le M|\xi|^2, \quad \xi \in R^n,\ \hat{x} \in \omega$$

$$a_{in}(\hat{x}) \equiv 0 \text{ for } i < n,\ a_{nn}(\hat{x}) \equiv 1;\ m, M = \text{const } > 0,\ p = \text{const } > 1$$

We denote

$$S(a,b) = \{x : \hat{x} \in \omega, a < x_n < b\},\ \sigma(a,b) = \{x : \hat{x} \in \partial\omega, a < x_n < b\}$$

The function $u(x)$ is called a weak solution of problem (1), (2), if for any $T > 0$ function $u(x) \in H^1(S(0,T))$, $u(x)$ is bounded in $S(0,T)$ and

$$-\int_{S(0,T)} \sum_{i,j=1}^{n} a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\frac{\partial \varphi}{\partial x_i}dx + \int_{S(0,T)} \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i}\varphi dx = \int_{S(0,T)} |u|^{p-1}u\varphi dx \tag{3}$$

for any function $\varphi \in H^1(S(0,T))$, $\varphi(\hat{x},0) = 0$, $\varphi(\hat{x},T) = 0$.

Many problems of mathematical physics lead one to consider the solutions of problem (1), (2) and to study the behaviour of the solutions at infinity (stationary states, travelling waves, homogenization, boundary layer problems, Saint-Venant's principle and so on). These problems are considered in many papers (see, for example, [1]-[5]). For linear equations such problems are investigated in [4].

We shall use the following propositions for linear second order equations.

1. Consider the problem

$$L(u) + q(x)u = \sum_{i,j=1}^{n} \frac{\partial f_i}{\partial x_j} + f_0 \quad \text{in } S(-\infty, +\infty), \ \frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \sigma(-\infty, +\infty) \qquad (4)$$

under condition

$$J_h(f_i) \equiv \int_{S(-\infty,+\infty)} e^{2hx_n} |f_i|^2 dx < \infty, \ i = 0, 1, \cdots, n \qquad (5)$$

with constant h such that the eigenvalue problem

$$\sum_{i,j=1}^{n-1} \frac{\partial}{\partial x_i}\Big(a_{ij}(\hat{x})\frac{\partial u}{\partial x_j}\Big) + \sum_{i=1}^{n-1} a_i(\hat{x})\frac{\partial u}{\partial x_i} - \lambda^2 u = 0 \quad \text{in } \omega \qquad (6)$$

$$\frac{\partial u}{\partial \gamma} = 0 \quad \text{on } \partial\omega \qquad (7)$$

has no eigenvalues λ with $J_m h = J_m \lambda$. Then there exists a constant $\varepsilon > 0$ such that if

$$|q(x)| < \varepsilon \quad \text{in } S(-\infty, +\infty) \qquad (8)$$

there exists a unique solution $u(x)$ of problem (4) and

$$T_h(u) \equiv \sum_{i=1}^{n} \int_{S(-\infty,+\infty)} \Big(\frac{\partial u}{\partial x_i}\Big)^2 e^{2hx_n} + \int_{S(-\infty,+\infty)} |u|^2 e^{2hx_n} dx \le C_h \sum_{i=1}^{n} J_h(f_i) \qquad (9)$$

(see [6]).

2. Assume that $q(x) \equiv 0$, $u(x)$ is a solution of problem (4), satisfying condition $T_{h_2}(u) < \infty$,

$$J_{h_1}(f_i) < \infty, \ J_{h_2}(f_i) < \infty, \ i = 0, 1, \cdots, n$$

and, in addition, in the strip $h_1 \le J_m\lambda \le h_2$ there is only one eigenvalue λ_0 of the problem (6), (7), $h_1 < J_m\lambda_0 < h_2$. Then

$$u(x) = \sum_{j=0}^{k} C_j x_n^j \phi_j(\hat{x}) e^{i\lambda_0 x_n} + u_1(x)$$

$$T_{h_1}(u_1) \le C \sum_{i=0}^{n} J_{h_1}(f_i), \ C, C_j = \text{const} \qquad (10)$$