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NONLINEAR SUPERPOSITION OF DELTA WAVES IN MULTI-DIMENSIONAL SPACE

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Abstract We study the behavior of the solution u^ε to the semilinear wave equation with initial data $a_i^\varepsilon + u_i (i = 1, 2)$ in multidimensional space, where u_i is a classical function and a_i^ε is smooth and converges to a distribution a_i as $\varepsilon \to 0$. In some circumstances one can prove the convergence of u^ε, and our results express a striking superposition principle. The singular part of the solution propagates linearly. The classical part shows the nonlinear effects. And, the limit of the nonlinear solution u^ε, delta wave, as the data become more singular is the sum of the two parts.

Key Words Semilinear wave equation; delta wave; superposition principle
Classification 35O85, 35L45, 35L60.

1. Introduction

In [1], J. Rauch & M. Reed studied the behavior of solutions, u^ε, to the strictly hyperbolic semilinear system in one space dimension

$$\begin{cases} Lu^\varepsilon = (\partial_t + A(x,t)\partial_x + B(x,t))u^\varepsilon = f(x,t,u^\varepsilon) \\ u^\varepsilon|_{t=0} = g + h^\varepsilon \end{cases}$$

where $g \in L^1$ is a classical function, h^ε is smooth and converges to a distribution μ as $\varepsilon \to 0$. The authors used the character of that system which can integrate along the characteristic in one space dimension sufficiently, and confirmed that the limit of solution u^ε as the data become more singular is the sum of the two parts — "singular" and "classical", denoted by σ and $\bar{u}$ respectively, where $\sigma \in \mathcal{D}'$, $\bar{u} \in C([0,T]; L^1)$. They call $\bar{u} + \sigma$ the solution of $Lu = f(t,x,u)$ with initial data $g + \mu$. The singular part of the solution propagates linearly. The classical part shows the nonlinear effects. In [2], there is a similar result to the nonlinear Klein-Gordon equation in one space dimension. However, in multidimensional space the problem as above becomes more difficult and complex. It can be illustrated by the wave equation. First, there isn't any estimate for $L^p \to L^p (p \neq 2)$, and furthermore, the support of the fundamental solution E_+ of the wave equation varies as dimensional changes. If n is odd and $n > 1$, both the support and singular support of E_+ are exactly equal to the boundary of the forward light cone. For all other values of n, the support is exactly equal to the forward light cone itself but the singular support is on the boundary of the forward light cone. As $n \to +\infty, E_+$ becomes less and less "regular". If $n = 1, E_+$ is bounded. In both cases $n = 2$ and $n = 3$, we know that $E+$ is a Radon measure. When $n = 2, E_+$ is absolutely continuous with respect to the Lebesgue measure, but not bounded. When $n = 3, E_+$ is a measure carried by the surface of the forward light cone. When $n = 4$, it ceases to be a measure. It is the distribution derivative of a measure carried by the forward light cone[3]. It is not difficult to see that there must be more additional assumptions on nonlinearity of f if one requires that the superposition principle holds.

We consider the following problems at first:

$$\begin{cases} Lu^\varepsilon = f(u^\varepsilon) \\ u^\varepsilon|_{t=0} = h^\varepsilon + u_0, \end{cases} \qquad \begin{cases} L\sigma^\varepsilon = 0 \\ \sigma^\varepsilon|_{t=0} = h^\varepsilon, \end{cases} \qquad \begin{cases} L\bar{u} = f(\bar{u}) \\ \bar{u}|_{t=0} = u_0 \end{cases}$$

where h^ε approaches delta function in the sense of distribution. Requiring that the limit of u^ε has superposition property, we must show that $v^\varepsilon = u^\varepsilon - \sigma^\varepsilon - \bar{u}$ converges to zero in some sense as $\varepsilon \to 0$. Let $u_0 \equiv 0$, and $f(0) = 0$, we have $\bar{u} \equiv 0$. σ^ε converges to the fundamental solution in the sense of distribution. So it is equivalent to proving that the solution of the following problem

$$\begin{cases} Lu^\varepsilon = f(u^\varepsilon) \\ u^\varepsilon|_{t=0} = h^\varepsilon \end{cases}$$

approaches fundamental solution in the sense of distribution as $\varepsilon \to 0$. This requires that $f(u^\varepsilon) \to 0$ in the sense of distribution.

As an example, we consider the following initial problem in three space dimensions

$$\begin{cases} \Box u^\varepsilon = f(u^\varepsilon) \\ u^\varepsilon|_{t=0} = \delta_\varepsilon(r) \\ u_t^\varepsilon|_{t=0} = 0 \end{cases}$$

where $\delta_\varepsilon(r)$ equals $1/\varepsilon^3$ for $r \le \varepsilon$, and equals 0 for $r > \varepsilon$. $r = (\Sigma_1^3 x_i^2)^{1/2}, x = (x_1, x_2, x_3) \in R^3$. Solving this initial problem with $f \equiv 0$, we have

$$ru^\varepsilon = \frac{1}{2}[(r+t)\delta_\varepsilon(r+t) + (r-t)\delta_\varepsilon(r-t)]$$

When $t > \varepsilon$, we have

$$u^\varepsilon = \begin{cases} 0, & |r-t| > \varepsilon \\ \dfrac{1}{2r\varepsilon^3}(r-t), & |r-t| \le \varepsilon \end{cases}$$