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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS WITH MIXED BOUNDARY CONDITIONS [1)]

Xue Ruying
(Dept. of Math., Hangzhou University, Hangzhou, 310028)
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Abstract In this paper, the existence of positive solutions for the mixed boundary problem of quasilinear elliptic equation

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + f(x,u), & u>0, \quad x\in\Omega \\ u|_{\Gamma_0} = 0, \quad \dfrac{\partial u}{\partial \vec{n}}\Big|_{\Gamma_1} = 0 \end{cases}$$

is obtained, where Ω is a bounded smooth domain in R^N, $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}_1, 2 \le p < N$, $p^* = \dfrac{Np}{N-p}$, Γ_0 and Γ_1 are disjoint open subsets of $\partial\Omega$.

Key Words Critical point theory; quasilinear elliptic equation; mixed boundary condition; isoperimetric constant.
Classifications 35J20, 35D05, 35J60.

1. Introduction

Let Ω be a bounded smooth domain in $R^N (N \ge 3)$ whose boundary $\partial\Omega$ is made of two manifolds Γ_0 and Γ_1, Γ_0 and Γ_1 have positive $(N-1)$-dimensional Hausdorff measures. In this paper we are concerned with the existence of positive solutions for quasilinear elliptic equation with mixed boundary conditions

$$\text{(P)} \quad \begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = |u|^{p^*-2}u + f(x,u), & u>0, \quad x\in\Omega \\ u|_{\Gamma_0} = 0, \quad \dfrac{\partial u}{\partial \vec{n}}\Big|_{\Gamma_1} = 0 \end{cases}$$

where $2 \le p < N, p^* = \dfrac{Np}{N-p}$; $f(x,u)$ is a lower-order perturbation of $|u|^{p^*-2}u$ in the sense that $\lim\limits_{u\to\infty} \dfrac{f(x,u)}{|u|^{p^*-2}u} = 0$; $\vec{n}$ is the outward unit normal to $\partial\Omega$.

Many satisfactory results on the existence of nontrivial solutions have been obtained in the cases $\Gamma_1 = \emptyset$ or $p^* < \dfrac{Np}{N-p}$ (see [1]–[6]), but few have been known in the case

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$\Gamma_1 \neq \emptyset$ and $p^* = \dfrac{Np}{N-p}$. There are two difficulties in the study of problem (P): one is that $p^* = \dfrac{Np}{N-p}$ is the limiting Sobolev exponent for the embedding $V^p(\Omega, \Gamma_1) \hookrightarrow L^{p^*}(\Omega)$, where $V^p(\Omega,\Gamma_1) = \{u \in H^{1,p}(\Omega) | u = 0 \text{ on } \Gamma_1\}$, the other is that the infimum

$$S(\Omega,\Gamma_1) = \inf\Big\{\int_\Omega |\nabla u|^p dx | u \in V^p(\Omega,\Gamma_1) - \{0\}, \quad \int_\Omega |u|^{p^*} dx = 1\Big\}$$

depends on Ω and Γ_1 (see [7]).

In this paper, we develop the methods in [1] and [5], and obtain the existence results for the problem (P). Our results are based on those of P.L. Lions, F. Pacella and M. Tricarico [7].

2. Preliminaries

Let C_N be the measure of the unit ball in R^N. As shown in [7], it is possible to associate with the set Ω an isoperimetric constant $Q(\Omega,\Gamma_1) \in [(NC_N^{1/N})^{-1}, \infty)$. Then there exists a number $\alpha \in (0,\pi]$ such that $Q(\Omega,\Gamma_1) = (N(\alpha_N)^{1/N})^{-1}$, where α_N is the measure of the unitary sector $\Sigma(\alpha,1), \Sigma(\alpha,R)$ denotes the sector of redius R and amplitude α defined by

$$\sum(\alpha,R) = \left\{x \in R^N \,\middle|\, \begin{array}{l} 0 \le |x| < R, \theta_i \in (0,\pi) \quad (1 \le i \le N-2), \theta_{N-1} \in (0,\alpha) \\ (\rho,\theta_1,\cdots,\theta_{N-1}) \text{ is the polar coordinates in } R^N \end{array}\right\}$$

$$\Gamma_0 = \Big\{x \in \partial\sum(\alpha,R) \| x| = R\Big\}, \quad \Gamma_1 = \partial\sum(\alpha,R) - \Gamma_0$$

Let ε_{α_N} be the class of open sets whose isoperimetric constants $Q(\Omega,\Gamma_1)$ are given by $(N(\alpha_N)^{1/N})^{-1}$. We have

Theorem 2.1[7] *If $\Omega \in \varepsilon_{\alpha_N}$ and $1 < p < N, p^* = \dfrac{Np}{N-p}$, we have*

$$\int_\Omega |\nabla u|^p dx \ge S(\alpha_N)\Big(\int_\Omega |u|^{p^*} dx\Big)^{\frac{p}{p^*}}, \quad u \in V^p(\Omega,\Gamma_1)$$

where

$$S(\alpha_N) = \left(\frac{B^{\frac{1}{p^*}}}{N\alpha_N^{1/N}}\right)^{-1}, \quad B = \Big(1-\frac{1}{p}\Big)^{p^*}\left\{\frac{\Gamma(N)}{\Gamma(\frac{N}{p})\Gamma(N+1-\frac{N}{p})}\right\}^{-\frac{p}{N}}$$

Moreover, the constant $S(\alpha_N)$ is achieved for $\Omega = \Sigma(\alpha,R)$ for any $R > 0$.

Define

$$S(\Omega,\Gamma_1) = \inf\Big\{\int_\Omega |\nabla u|^p dx | u \in V^p(\Omega,\Gamma_1), \int_\Omega |u|^{p^*} dx = 1\Big\} \tag{2.1}$$

$$\|u\| = \Big(\int_\Omega |\nabla u|^p dx\Big)^{\frac{1}{p}}, \quad u \in V^p(\Omega,\Gamma_1) \tag{2.2}$$