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THE BOUNDEDNESS FOR GENERALIZED SOLUTIONS OF QUASILINEAR ELLIPTIC EQUATIONS

Liang Xiting
(Dept. of Math., Zhongshan Univ.)
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Abstract Under the appropriate conditions on u, the generalized solution of the elliptic equation

$$\int_G \{\nabla v \cdot A(x,u,\nabla u) + vB(x,u,\nabla u)\}dx = 0, \quad \forall v \in \overset{\circ}{W}{}_p^1(G) \cap L_\infty(G)$$

for which even the natural growth condition $p(1-1/p^*) < \gamma < p$ is permitted, the local and global boundedness of u are proved.

Key Words Elliptic equation; critical exponent; natural growth condition; generalized solution; boundedness; Hölder continuity.

Classifications 35B35; 35D10; 35J20.

Let G be a bounded domain in the n-dimensional Euclidean space E^n. Let $p > 1$ and $W_p^1(G)$ and $\overset{\circ}{W}{}_p^1(G)$ be the usual Sobolev spaces. Consider on G the following elliptic equation

$$\int_G \{\nabla v \cdot A(x,u,\nabla u) + vB(x,u,\nabla u)\}dx = 0, \quad \forall v \in \overset{\circ}{W}{}_p^1(G) \cap L_\infty(G) \tag{1}$$

where $A(x,u,\xi)$ and $B(x,u,\xi)$ are defined on $G \times E^1 \times E^n$, continuous in u and ξ for fixed x and measurable in x for fixed u and ξ, respectively. Moreover, suppose A and B satisfy the following structure inequalities:

$$\begin{aligned} &\nabla u \cdot A(x,u,\nabla u) \geq |\nabla u|^p - \kappa|u|^q - f_0(x) \\ &|A(x,u,\nabla u)| \leq \kappa_1|\nabla u|^{p-1} + \kappa|u|^{q/p'} + f_1(x) \\ &|B(x,u,\nabla u)| \leq c(x)|\nabla u|^\gamma + \kappa|u|^{q-1} + f_2(x) \end{aligned} \tag{2}$$

where $1 < p < n$, $\kappa \geq 0$, $\kappa_1 \geq 1$, $q = p^* = np/(n-p)$, $1/p' + 1/p = 1$ and $p - 1 \leq \gamma < p$ are all constants;

$$c(x) \in L_r(G), r = n \text{ as } \gamma = p-1 \text{ and } r > n/(p-\gamma) \text{ as } \gamma > p-1 \tag{3}$$

$$f_i(x) \in L_{s_i}(G)(i = 0,1,2), s_0, s_2 > n/p \text{ and } s_1 > n/(p-1) \tag{4}$$

It is well known that the exponent q of u can not exceed the critical Sobolev exponent p^* in the structure inequalities (2) as one looks for a solution of the equation (1) in $W_p^1(G), 1 < p < n$. At the same time the growth order γ of ∇u can not exceed p in B. For the case of $q < p^*$ and $\gamma < p(1-1/p^*)$ there have been sufficient investigations, see [1, 2]. But the case of q arriving at the critical Sobolev exponent p^* is more and more attracted nowadays. In [3], Zhu and Yang prove for the case of $q = p^*$ and $\gamma = p(1-1/p^*)$ the boundedness of the generalized solutions and then the regularity. In the case of $\gamma = p$ one can only look for the solutions in bounded functional classes, otherwise the local uniqueness theorem will be failed. In the case $p = 2$, it is pointed out by Giaquinta in [4] that even the natural growth condition $2(1-1/2^*) < \gamma < 2$ is permitted the local Hölder continuity will be proved for the solution u of the equation

$$\int_G \nabla v \cdot \nabla u dx = \int_G v|\nabla u|^\gamma dx, \quad \forall v \in \overset{\circ}{W}{}_2^1(G) \cap L_\infty(G)$$

if only $u \in W_2^1(G) \cap L_{\frac{n(\gamma-1)}{2-\gamma}}(G)$. Now let

$$t = \frac{n(\gamma+1-p)}{p-\gamma} \text{ as } r = \infty; \; t = \frac{nr(\gamma+1-p)}{r(p-\gamma)-n} \text{ as } \frac{n}{p-\gamma} < r < \infty$$

It is proved in this paper that the solution $u \in W_p^1(G) \cap L_t(G)$ of the equation (1) satisfying the structure conditions (2) is locally bounded. If the boundary ∂G of the domain G is restricted so that the condition (28) (see below) is fulfilled, then the solution $u \in \overset{\circ}{W}{}_p^1(G) \cap L_t(G)$ of the equation (1) is globally bounded on G, furthermore, u is uniformly Hölder continuous on G.

Lemma[5] *Let $B(\rho) = \{|x| < \rho\}$ and $u \in W_p^1(B(\rho))$, $1 < p < n$. Suppose $u = 0$ on a positive measure set S in $B(\rho)$. Let $\eta(x) = \eta(|x|)$ be a nonincreasing and continuous function of $|x|$ with values from $[0,1]$ and satisfy $\eta(x) = 1$ as $x \in S$. Then holds*

$$\Big(\int_{B(\rho)} \Big||u(x)|\eta(x)\Big|^{p^*} dx\Big)^{1/p^*} \le C\Big(n, p, \frac{\rho^n}{\text{mes } S}\Big)\Big(\int_{B(\rho)} \Big|\nabla u(x)|\eta(x)\Big|^p dx\Big)^{1/p}$$

Theorem 1 *Suppose the conditions (2)–(4) are fulfilled and*

$$u \in W_p^1(G) \text{ as } \gamma = p-1 \text{ and } u \in W_p^1(G) \cap L_t(G) \text{ as } p-1 < \gamma < p$$

satisfy the equation (1), then u is bounded locally in G.

Proof We assume $B(2\rho) \subset G$. Let $\rho \le \rho_1 < \rho_0 \le 2\rho$ and $\varsigma(x) = \varsigma(|x|)$ be a piecewise linear and continuous function of $|x|$ satisfying $\varsigma(x) = 1$ as $|x| \le \rho_1$ and $\varsigma(x) = 0$ as $|x| \ge \rho_0$. Then $|\nabla\varsigma(x)| \le (\rho_0 - \rho_1)^{-1}$. Let $\tau = p^2/(p-1)$, $h > k \ge 0$ and $(u-k)^+ = \max(u-k, 0)$. We take $v = \varsigma^\tau[(u-k)^+ - (u-h)^+] \in \overset{\circ}{W}{}_p^1(G) \cap L_\infty(G)$ as