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THRESHOLD BEHAVIOR FOR SECOND INITIAL BOUNDARY VALUE PROBLEM OF *F-N* EQUATIONS

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Abstract Under biologically reasonable assumptions the threshold behavior for second initial boundary value problem of *F-N* equation is proved.

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In this note we consider the second initial boundary value problem of *F-N* equations in bounded interval as follows:

$$\begin{cases} v_t = v_{xx} + f(v) - u, & 0 < x < L \\ u_t = \sigma v - \gamma u, & t > 0 \\ t = 0 : v = 0, & u = 0 \\ x = 0 : v_x = h(t); & x = L : v_x = 0 \end{cases} \tag{1}$$

where v is electrical potential and $h(t)$ represents the stimulus given in the form of electrical current at the end $x = 0$. The threshold behavior, which is very interesting from the point of view of neurobiology, in the term of mathematics is to show that if the strength of stimulus $\|h\|_C$ or the lasting time T_0 of stimulus is small then the solution to (1) decays to zero as $t \to \infty$.

Although there are many works on threshold behavior for infinite axon or semi-bounded axon ([1]–[4]), very little was known for bounded axon. But a real nerve has only finite extent.

The biologically reasonable assumptions on (1) are the following:

$$f(v) \in C^2, \quad f(0) = 0, \quad f'(0) = -a < 0 \tag{2}$$

$$\sigma, \gamma \text{ are positive constants}, \quad \sigma > 0, \quad \gamma > 0 \tag{3}$$

$$h(t) \in C^1, \quad h(0) = 0, \quad h(t) = 0 \text{ for all } \quad t \geq T_0 > 0 \tag{4}$$

Throughout this note we use the following notation:

$$U=\begin{pmatrix}v\\u\end{pmatrix}, \|U(t)\|_C=\|v(t)\|_{C([0,L])}+\|u(t)\|_{C([0,L])}$$

$$\|h\|_C=\|h\|_{C([0,T_0])},\qquad \|U(t)\|^2_{L^2}=\|v(t)\|^2_{L^2([0,L])}+\|u(t)\|^2_{L^2([0,L])}$$

The main results of this paper are the following:

Theorem 1 *Under assumptions (2)–(4) for problem (1) we have*

(i) *For any fixed $T_0>0$ there exists a small constant $\varepsilon_{T_0}>0$ such that if $\|h\|_C\le\varepsilon_{T_0}$ then problem (1) admits a unique global smooth solution $U(x,t)$ and $\|U(t)\|_C$ exponentially decays to zero.*

(ii) *For any $N>0$ there exists a small constant r_N such that if $\|h\|_C\le N$, $T_0\le r_N$, then problem (1) admits a unique global smooth solution $U(x,t)$ and $\|U(t)\|_C$ exponentially decays to zero.*

The proof of Theorem 1 consists of several lemmas.

Lemma 1 *For any $U_0\in C^2$ there exists a constant $\delta>0$ depending on $\|h\|_C$, $\|U_0\|_C$, T_0 such that problem*

$$\begin{cases} v_t=v_{xx}+f(v)-u, & 0<x<L\\ u_t=\sigma v-\gamma u, & t>t_0,\\ t=t_0: v=v_0(x), & u=u_0(x)\\ x=0: v_x=h(t); & x=L: v_x=0\\ h(t_0)=v_0'(0), & v_0'(L)=0\end{cases}\tag{5}$$

has a unique smooth solution in $[0,L]\times[t_0,t_0+\delta]$ and

$$\|U(t)\|_C\le K_1(\|U_0\|_C+T_0\|h\|_C)\tag{6}$$

Proof The solution to (5) can be written as follows

$$\begin{cases} v(x,t)=\omega(x,t)+\sum_{n=0}^{\infty}\int_{t_0}^{t}[f(v(x,\tau))-u(x,\tau),X_n]e^{-\lambda_n(t-\tau)}d\tau X_n\\ u(x,t)=\sigma\int_{t_0}^{t}e^{-\gamma(t-\tau)}v(x,\tau)d\tau-e^{-\gamma t}u_0(x)\end{cases}\tag{7}$$

where $\omega(x,t)$ is the solution to the following problem:

$$\begin{cases} \omega_t=\omega_{xx}, \ t>t_0,\ 0<x<L\\ t=t_0:\omega=v_0(x)\\ x=0:\omega_x=h(t);\quad x=L:\omega_x=0\end{cases}\tag{8}$$