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Cauchy Problem for a Fractional Parabolic Equation with the Advection

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Abstract. In this paper, we give an explicit formula of the solution to Cauchy problem of a fractional parabolic equation with the advection, and then prove the solvability of Cauchy problem and further Schauder-type regularity of the solution under appropriate conditions on the initial value and the right-hand side term.

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1 Introduction

The fractional Laplacian operators and integro-differential operators have attracted increasing attention over the last ten years. The operators of this type arise in a natural way in many applications such as continuum mechanics, phase transition phenomena, population dynamics, image process, game theory and Lévy processes.

The fractional Laplacian operator $(-\Delta)^s$ $(0<s<1)$ related to Lévy process in the probabilistic approach was studied in [1–4]. The operator $(-\Delta)^s$ can be defined by using the Fourier transform as

$$\big((-\Delta)^s u\big)^\wedge(\xi)=|\xi|^{2s}\hat{u}(\xi)$$

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for any rapidly decaying C^∞-function u in the Schwartz space $\mathbf{S}(\mathbb{R}^N)$, where the Fourier transform $\widehat{u}(\xi)$ of the function $u(x)$ is defined by

$$\widehat{u}(\xi)=\frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N}u(x)e^{-ix\cdot\xi}\mathrm{d}x.$$

A more useful classical formula is given by

$$(-\Delta)^s u(x)=C_{N,s}\left(\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y\right)=C_{N,s}\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{N+2s}}\mathrm{d}y,$$

where

$$C_{N,s}=\frac{s4^s\Gamma(\frac{N}{2}+s)}{\pi^{\frac{N}{2}}\Gamma(1-s)} \tag{1.1}$$

is the normalization constant ([4,5]).

For $0<s<1$, the parabolic integro-differential equation

$$u_t(x,t)+(-\Delta)^s u(x,t)=0, \quad \text{in } \mathbb{R}^N\times(0,+\infty)$$

is a natural generalization of the heat equation. Droniou and Imbert [6] proved that there is a unique viscosity solution of

$$u_t+H(\nabla_x u)+(-\Delta)^s u=0$$

in $W^{1,\infty}$ (i.e. Lipschitz) for any value of $s\in(0,1)$ if the initial data $u(\cdot,0)$ is Lipschitz and H is locally Lipschitz. Silvestre [7] obtained the Hölder continuity of the solutions of

$$u_t(x,t)+\mathbf{b}(x,t)\cdot\nabla_x u(x,t)+(-\Delta)^s u(x,t)=f(x,t) \tag{1.2}$$

for $s\in[\frac{1}{2},1)$ and any bounded vector field $\mathbf{b}$, and for $s\in(0,\frac{1}{2})$ and any C^{1-2s}-Hölder continuous vector field $\mathbf{b}$, respectively.

The critical case $s=\frac{1}{2}$ in (1.2) attracts much attention since the second term and the third term on the left-hand side are of order one and their contributions are balanced at every scale. Silvestre [8] investigated the existence of classical $C^{1,\alpha}$-solutions for the Hamilton-Jacobi equation with critical fractional diffusion

$$u_t+H(\nabla_x u)+(-\Delta)^{\frac{1}{2}}u=0 \quad \text{or} \quad u_t+\mathbf{b}\cdot\nabla_x u+(-\Delta)^{\frac{1}{2}}u=0$$

by using comparison principles provided that $\mathbf{b}$ is divergence free. Caffarelli and Vasseur [9] obtained a classical solution $u\in C^{1,\beta}([t_0,+\infty)\times\mathbb{R}^N)$ for

$$u_t+\mathbf{b}\cdot\nabla_x u+(-\Delta)^{\frac{1}{2}}u=0, \qquad \mathrm{div}\,\mathbf{b}=0$$

by using De Giorgi's approach for any $\beta\in(0,1)$ and $t_0>0$ provided that $\mathbf{b}$ belongs to the BMO class and the weak solution $u\in L^\infty(0,\infty;L^2)\cap L^2(0,\infty;H^{\frac{1}{2}})\cap L^\infty([t_0,\infty)\times\mathbb{R}^N)\cap$