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Gradient Estimates and Liouville-Type Theorems for a Nonlinear Elliptic Equation

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Abstract. In this paper, we consider the following nonlinear elliptic equation

$$\triangle_f u + hu^\alpha = 0$$

on the complete smooth metric space $(R^n, g_0, e^{-f}dv_{g_0})$, where g_0 is the Euclidean metric on R^n and $f = |x|^2/4$. We prove gradient estimates and Liouville-Type theorems for positive solutions of the above equation.

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Key Words: Gradient estimate; Liouville-Type theorem; shrinking gradient Ricci soliton.

1 Introduction

Recall that a triple (M,g,f) is called a shrinking gradient Ricci soliton (c.f. [1]) if

$$Ric + \nabla^2 f = \lambda g$$

for some positive constant λ, where (M,g) is a Riemannian manifold and f is a smooth function on M. We say that the gradient soliton is complete if both (M,g) and the vector field $\nabla_g f$ is complete. We call the function f a potential function.

The Bakry-Emery Ricci tensor on the Riemannian manifold (M,g) is defined by

$$Ric_f = Ric + \nabla^2 f$$

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for some smooth function f on M. Thus if $Ric_f=\lambda g$ for some positive constant λ, then (M,g,f) is a shrinking gradient Ricci soliton.

The f-Laplacian operator is defined by

$$\triangle_f=\triangle-\nabla f\cdot\nabla.$$

If f is constant, the f-Laplacian operator is reduced to the classical Laplacian. If a smooth function u on M satisfies $\triangle_f u=0$ $(\geq 0,\ \leq 0)$, we call it f-harmonic (f-subharmonic, f-superharmonic). For a positive f-harmonic function, its gradient estimate and Liouville theorem have been studied by many authors. Wei and Wylie [2] proved that any positive f-harmonic function with some growth conditions must be constant if $Ric_f\geq K>0$. Brighton [3] proved that a positive bounded f-harmonic function with $Ric_f\geq 0$ is constant. For a local Cheng-Yau's gradient estimate, Wu [4] obtained it for positive f-harmonic functions with $Ric_f\geq -(n-1)K$ and $|\nabla f|\leq\theta$. Chen and Chen [5] also proved the same gradient estimate for positive f-harmonic functions with another condition $Ric\geq-(n-1)H$. Munteanu and Sesum [6] applied the De Giorgi-Nash-Moser theory to get a global gradient estimate for a positive f-harmonic function and proved that a positive f-harmonic function with sublinear growth of f on the metric space is constant if $Ric_f\geq 0$. Li [7] applied probabilistic arguments to give an alternative proof of Brighton's gradient estimate and Liouville theorem for positive f-harmonic functions. Wu [8] proved a Liouville property for any f-harmonic function with polynomial growth on a complete noncompact smooth metric measure space on which any diameter of geodesic sphere has sublinear growth and whose Bakry-Emery Ricci curvature satisfies a quadratic decay lower bound, i.e., $Ric_f\geq -cr^{-2}$.

In this paper, we are interested in the following nonlinear elliptic equation

$$\triangle u-\frac{r}{2}\nabla r\cdot\nabla u+hu^{\alpha}=0 \tag{1.1}$$

on the space (R^n, g_0, $e^{-|x|^2/4}dv_{g_0}$) which is a complete smooth metric space and is a shrinking gradient Ricci soliton. Here g_0 is the Euclidean metric on R^n. For a quasi-harmonic function u on R^n, u satisfies the equation (1.1) for $h=0$. Li and Wang [9] derived gradient estimates for positive quasi-harmonic functions and showed that there is no nonconstant positive quasi-harmonic function on R^n with polynomial growth. Zhu and Wang [10] showed that there is neither a nonconstant positive quasi-harmonic function nor a nonconstant $L^p(R^n,ds^2)$ $(p>\frac{n}{n-2},\ n\geq 3)$ quasi-harmonic function, where $ds^2=e^{-|x|^2/2(n-2)}g_0$. But for all $1\leq p\leq n/(n-2)$, there exists a nonconstant quasi-harmonic function in $L^p(R^n,ds^2)$ $(n\geq 3)$. Ge and Zhang [11] proved that there doesn't exist a nonconstant positive f-harmonic function on the complete gradient shrinking Ricci solitons. They also obtained L^p $(p\geq 1$ or $0<p\leq 1)$ Liouville theorems on the complete gradient shrinking Ricci solitons. We derive the similar results for positive solutions of equation (1.1).