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WEAK SOLUTIONS OF SECOND ORDER PARABOLIC EQUATIONS IN NONCYLINDRICAL DOMAINS

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Abstract

Penalization method is used to get the existence of the weak solutions of parabolic equations in noncylindrical domains. The asymptotic decay and the existence and uniqueness of the periodic solutions are obtained as well.

1. Introduction

Let D be a bounded domain in $R^n \times (0, T)$. Let L be a second order parabolic differential operator of the following form:

$$Lu \equiv u_t - \{ (a_{ij}u_{x_j} + a_i u)_{x_i} - b_i u_{x_i} - a_0 u\} \tag{1. 1}$$

where, the coefficients of L belong to some suitable spaces (see section 2). We consider the following problem:

$$\begin{cases} Lu = (f_i)_{x_i} - f_0 & \text{in} \quad \mathscr{D}'(D) \\ u = \psi, & \text{on} \quad \partial_p D \end{cases} \tag{1. 2}$$

where $\partial_p D$ is the parabolic boundary of D, the meaning of it will be specified later, and f_i, f_0 belong to some spaces (see section2).

For the case $D = \Omega \times (0, T)$, namely, D is a cylindrical bounded domain, the problem has been substantially discussed, see, for example, [6], [9], [12], and the references cited therein. For the case D is ono-cylindrical, the problem has also been studied by many authors. In [9], the existence and uniqueness of a classical solution to (1. 2) with L of non-divergence form was obtained under some smoothness conditions on data and $\partial_p D$. Under an abstract framework, the existence of a weak solution of (1. 2) was obtained in [11], but the uniqueness result was rather restrictive. See also [13]. In [5], the Perron's method was used to get the existence and uniqueness of a generalized solution to (1. 2). Recently, the problem has been discussed under the framework of intermediate Schauder theory [10].

In this paper, we use the penalization method to establish the existence of the week solution of (1. 2). By this method, we obtain an estimate of the weak solution, the stability of the weak solution with respect to the data, the asymptotic decay of the weak solution and the existence and uniqueness of the periodic solution of (1. 2). In doing the above, we only assume some very weak conditions on the parabolic boundary $\partial_p D$ of D.

The penalization method is frequently used in many free-boundary problems, see [2], [7], for example.

2. Preliminaries

Let Ω be a bounded domain in R^n with smooth boundary $\partial\Omega$. We denote $\Omega_T = \Omega \times (0, T)$, $\Omega_\infty = \Omega \times (0, \infty)$ and $\partial_p \Omega_T = (\partial\Omega \times [0, T]) \cup (\Omega \times \{0\})$. The spaces $L_{q,r}(\Omega_T)$, $W_2^{1,1}(\Omega_T)$, $V_2^{1,0}(\Omega_T)$ and $\overset{\circ}{W}{}_2^{1,1}(\Omega_T)$, $\overset{\circ}{V}{}_2^{1,0}(\Omega_T)$ etc. and the norms $\|\cdot\|_{q,r,\Omega_T}$, $\|\cdot\|_{\Omega_T}$, etc. are defined as in [9]. Let $D_\infty \subset \Omega_\infty$ be a region such that for all $t \in [0, \infty)$, $B_t \equiv D_\infty \cap (\Omega \times \{t\})$ is a region in $\Omega \times \{t\}$. (For simplicity, we assume that B_t is simply connected for all $t \in [0, \infty)$.) We denote $D_T = D_\infty \cap \Omega_T$. Also, without loss of generality, we assume that

$$\{(x, t) \in R^n \times (0, T) \mid d((x, t), D_T) \leq 1\} \subseteq \Omega_T \equiv \Omega \times (0, T), \ \forall T > 0$$

where $d(\cdot, \cdot)$ is the Euclidean distance in R^{n+1}. In order to define the parabolic boundary of D_T, we need the following.

Definition 2. 1 Let $(x_0, t_0) \in D_T$ and $(x_1, t_1) \in \overline{D_T}$. We say that $(x_0, t_0) \succ (x_1, t_1)$, if there exists a polygonal path $\{(x(a), t(a)) \mid 0 \leq a \leq 1\} \subset D_T$, such that

$$\begin{cases} (x(0), t(0)) = (x_0, t_0), & (x(1), t(1)) = (x_1, t_1) \\ t(a) \quad \text{is decreasing} \end{cases}$$

We set

$$P_{D_T} = \{(x, t) \in \partial D_T \mid \exists \ (x_j, t_j) \to (x, t), \ (x_j, t_j) \in D_T, \ (x_j, t_j) \succ (x, t)\}$$

Then, we define the parabolic boundary $\partial_p D_T$ of D_T to be the following

$$\partial_p D_T = \overline{P_{D_T}}$$

This definition is adapted from [5]. Next, we define the lateral boundary $\partial_L D_T$ of D_T and the bottom $\partial_B D_T$ of D_T as follows:

$$\partial_L D_T = \overline{\{(x, t) \in \partial D_T \mid \exists \ x_j \to x, \ (x_j, t) \in D_T\}}$$

$$\partial_B D_T = \partial_p D_T \backslash \partial_L D_T$$

For simplicity, we denote $\partial_B D_T = B$. Since we assume every B_t is simply connected, we have $B = B_0 \subset \Omega \times \{0\}$.

Remark 2. 2 If ∂D_T is smooth, then $\partial_B D_T \equiv B$ is the flat part of ∂D_T with outward