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THE QUASILINEAR ELLIPTIC EQUATION ON UNBOUNDED DOMAIN INVOLVING CRITICAL SOBOLEV EXPONENT

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1. Introduction

This paper is concerned with the existence of nontrivial solution for the quasilinear elliptic equations

$$-\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)+a(x)|u|^{p-2}u=|u|^{p^*-2}u+f(x,u) \qquad (1.1)$$

with zero-Dirichlet condition on R^N, where $p^*=\frac{Np}{N-p}$, $f(x,0)=0$ and $f(x,u)$ is a lower—order perturbation of $|u|^{p^*-2}u$ in the sense that $\lim_{u\to\infty}f(x,u)/|u|^{p^*-2}u=0$.

The translation invariance of R^N is a typical difficulty in the study of elliptic equations on R^N. Indeed the translation invariance causes Sobolev embeddings lose compactness. In [1], we have given an approach to get some existence results for $p^*<\frac{Np}{N-p}$. In fact, the method in [1] is reducing the problems to local ones to gain some kinds of compactness. Note that $p^*=\frac{Np}{N-p}$ is the limiting Sobolev exponent for the embedding $W_0^{1,p}(\Omega)\subset L^{p^*}(\Omega)$, where Ω is a domain in R^N. This embedding is not compact even if it is a bounded domain. This shows that it is still difficult even if the problem is a local one. When $p=2$, Brezis & Nirenberg [2] have gained some existence results of equations (1.1) on bounded domains. But the method in [2] does not work anymore for quasilinear ones (i. e. $p>2$ case). The main difficult is the loss of weak continuity of $A_i(u)=|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}$ in $W_0^{1,p}(\Omega)$ and this is crucial for quasilinear equations. In [3], for bounded domains, we have overcome this difficult by a new approach.

In this paper we develop the methods in [1] and [3] and get the existence results for the quasilinear equations (1.1) on unbounded domain R^N. Our methods are based on the concentration-compactness principles due to P. L. Lions ([4], [5], [6]).

2. Preliminaries

We consider the problem

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2} \frac{\partial u}{\partial x_i}\right) + a(x)\,|u|^{p-2}u = |u|^{p^*-2}u + f(x, u) \\ u \in W^{1,p}(R^N),\ u \not\equiv 0 \end{cases} \tag{2.1}$$

where $p^* = \frac{Np}{N-p}$, $N > p \geq 2$.

Suppose $a(x)$, $f(x, u)$ be continuous functions satisfying following conditions:

(a) $a(x) \geq 0,\ \forall x \in R^N,\ a(x) \to \bar{a} > 0$ as $|x| \to +\infty$.

(b) $\lim_{t\to 0} f(x,t)/|t|^{p-2}t = 0,\ \lim_{t\to\infty} f(x,t)/|t|^{p^*-2}t = 0$, uniformly in x.

(c) $\frac{1}{p} t f(x,t) \geq F(x,t) = \int_0^t f(x,s)\,ds,\ \forall x \in R^N,\ t \in R$.

(d) $f(x,t) \underset{|x|\to+\infty}{\to} \bar{f}(t)$ for t bounded uniformly.

The energy functional of the problem (2.1) is

$$I(u) = \frac{1}{p}\int_{R^N} (|\nabla u|^p + a(x)\,|u|^p)\,dx - \frac{1}{p^*}\int_{R^N} |u|^{p^*}dx - \int_{R^N} F(x, u)\,dx,\ u \in W^{1,p}(R^N) \tag{2.2}$$

where $W^{1,p}(R^N)$ with the norm:

$$\| u \| = \left[\int_{R^N} (|\nabla u|^p + a(x)\,|u|^p)\,dx\right]^{\frac{1}{p}}.$$

Definition 2.1 A sequence $\{u_n\} \subset W^{1,p}(R^N)$ is called tight, if $\forall \varepsilon > 0, \exists\ R > 0$, such that $\int_{\{|x|\geq R\}} (|\nabla u_n|^p + |u_n|^p)\,dx < \varepsilon$, for all n.

Definition 2.2 For $c \in R$, a sequence $\{u_n\}$ in $W^{1,p}(R^N)$ is called $(PS)_c$ sequence, if

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{in } (W^{1,p}(R^N))^*$$

For fixed $c \in R$, if every $(PS)_c$ sequence is relative compact in $W^{1,p}(R^N)$, we called functional I satisfying $(PS)_c$ condition.

Let

$$I^\infty(u) = \frac{1}{p}\int_{R^N} (|\nabla u|^p + \bar{a}|u|^p)\,dx - \frac{1}{p^*}\int_{R^N} |u|^{p^*}dx - \int_{R^N} \bar{F}(u)\,dx,\quad u \in W^{1,p}(R^N) \tag{2.3}$$