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A Free Boundary Problem of a Semilinear Combustible System with Higher Dimension and Heterogeneous Environment

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Abstract. In this paper, we investigate a free boundary problem of a semilinear combustible system with higher dimension and heterogeneous environment. Such a problem is usually used as a model to describe heat propagation in a two-component combustible mixture in which the free boundary is described by Stefan-like condition. For simplicity, we assume that the environment and solutions are radially symmetric. We use the contraction mapping theorem to prove the local existence and uniqueness of the solution. Also we study the blowup property and the long time behavior of the solution. Our results show that when $pq>1$ blowup occurs if the initial datum is large enough and the solution is global and slow, whose decay rate is at most polynomial if the initial value is suitably large, while when $p>1$, $q>1$ there is a global and fast solution, which decays uniformly at an exponential rate if the initial datum is small.

AMS Subject Classifications: 35K20, 35R35, 92B05

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1 Introduction

In this paper, we consider the following free boundary problem with higher dimension

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and heterogeneous environment

$$
\begin{cases}
u_t - d\Delta u = v^p, & t>0,\ 0<r<h(t),\\
v_t - d\Delta v = u^q, & t>0,\ 0<r<h(t),\\
u_r(t,0) = v_r(t,0) = 0, & t>0,\\
u(t,h(t)) = v(t,h(t)) = 0, & t>0,\\
h'(t) = -\mu(u_r(t,h(t)) + \rho v_r(t,h(t))), & t>0,\\
h(0) = h_0,\ u(0,r) = u_0(r),\ v(0,r) = v_0(r), & 0\le r\le h_0.
\end{cases}
\tag{1.1}
$$

This problem is usually used as a model to describe heat propagation in a two-component combustible mixture. here $x=h(t)$ is moving boundary to be determined, $h_0>0$, $p\ge 1$, $q\ge 1$, d, μ and ρ are positive constants, and the initial function u_0 and v_0 satisfy

$$
\begin{cases}
u_0\in C^2([0,h_0]),\ \ v_0\in C^2([0,h_0]),\\
u_0'(0) = u_0(h_0) = v_0'(0) = v_0(h_0) = 0 \ \text{ and }\ u_0>0,\ v_0>0 \ \text{ in }\ (0,h_0).
\end{cases}
\tag{1.2}
$$

The equation governing the free boundary, $h'(t)=-\mu(u_r(t,h(t))+\rho v_r(t,h(t)))$, is a special case of the well-known Stefan condition, which was given by Josef Stefan in his papers appeared in 1889. The original Stefan problem treats the formation of ice in the polar seas. Until now, the Stefan condition has been used in the modeling of a number of applied problems, for example, it was used to describe the melting of ice in contact with water [1], in the modeling of oxygen in the muscle [2], and in wound healing [3] and tumor growth [4–6]. There is a vast literature on the Stefan problem, and some important recent theoretical advances can be found in [7].

Similar free boundary model can also describe ecological dynamics. We can refer to several earlier papers, for example, [8–14]. In [11], Du and Lin may be the first attempt to use the Stefan condition in the study of the spreading of populations. They proposed the following free boundary model

$$
\begin{cases}
u_t - du_{xx} = u(a-bu), & t>0,\ 0<x<h(t),\\
u_x(t,0) = 0,\ u(t,h(t)) = 0, & t>0,\\
h'(t) = -\mu u_x(t,h(t)), & t>0,\\
h(0) = h_0,\ u(0,x) = u_0(x), & 0\le x\le h_0.
\end{cases}
\tag{1.3}
$$

They gave a spreading-vanishing dichotomy for the solution of (1.3). Furthermore, they showed that when spreading occurs, for large time, the expanding front moves at a constant speed.

A corresponding work in a fixed domain with Dirichlet boundary condition can be