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A Regularity Criterion for the 3D Incompressible Density-Dependent Navier-Stokes-Allen-Cahn Equations

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Abstract. In this paper, we study the initial boundary problem for 3D incompressible density-dependent Navier-Stokes-Allen-Cahn equations, and give a regularity criterion for local strong solutions. Our result refines the blow-up criterion in [1].

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1 Introduction

In this paper, we are interested in the incompressible density-dependent Navier-Stokes-Allen-Cahn equations, which describes the motion of a mixture of two incompressible viscous fluids (see [1–3]):

$$\partial_t\rho+\nabla\cdot(\rho u)=0, \tag{1.1}$$
$$\partial_t(\rho u)+\nabla\cdot(\rho u\otimes u)+\nabla p=\nabla\cdot(2\eta(\chi)Du)-\delta\nabla\cdot(\nabla\chi\otimes\nabla\chi), \tag{1.2}$$
$$\nabla\cdot u=0, \tag{1.3}$$
$$\partial_t(\rho\chi)+\nabla\cdot(\rho u\chi)=-\mu, \tag{1.4}$$
$$\rho\mu=-\delta\Delta\chi+\rho\frac{\partial f}{\partial\chi}, \tag{1.5}$$

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for $(x,t)\in\Omega\times(0,+\infty)$, where $\Omega\subset\mathbb{R}^3$ is a bounded domain with smooth boundary, $\rho\in\mathbb{R}^+$, $u=(u_1,u_2,u_3)\in\mathbb{R}^3$, $p\in\mathbb{R}$ and $\chi\in\mathbb{R}$ denote the total density, the velocity field in the mixture, the pressure and the phase of the fluid components, respectively. $Du=\frac{1}{2}(\nabla u+\nabla u^T)$, μ is the chemical potential, $\eta(\chi)>0$ is called the viscosity coefficient of the mixture, δ is a positive constant corresponding to the width of the interface, and the free energy function satisfies the double-well structure $f(\chi)=\frac{1}{\delta}(\frac{\chi^4}{4}-\frac{\chi^2}{2})$. Notice that when χ is constant, the system reduces to the incompressible Navier-Stokes system and when ρ and χ are constants, the system reduces to the classical incompressible Navier-Stokes equations.

The system (1.1)-(1.5) is supplemented with the following Dirichlet boundary condition on the velocity and Neumann boundary condition on the phase-change variable

$$\left(u,\frac{\partial\chi}{\partial\nu}\right)\bigg|_{\partial\Omega}=(0,0),\qquad t\geq 0, \tag{1.6}$$

where ν is the unit outward normal vector of $\partial\Omega$, and the following initial conditions

$$(\rho,u,\chi)\Big|_{t=0}=(\rho_0,u_0,\chi_0),\qquad x\in\Omega. \tag{1.7}$$

Now we would like to mention some results on this system and its related model. For the compressible system, there are only a few theoretical available results. Feireisl et al. [4] dealt with questions of solvability and established the existence of global-in-time weak solutions without any restriction on the size of initial data. Ding et al. [5] proved the existence and uniqueness of global classical solution, the existence of weak solutions, and unique strong solution of the isentropic model in one-dimensional space for initial data without vacuum states. Kotschote [6] showed the existence and uniqueness of local strong solutions for non-isothermal model with arbitrary initial data. However, these results mentioned above are for the initial boundary value problem. For the incompressible system, the models have been extensively studied. Abels [7] obtained the existence of weak solutions with differential densities, while Abels [8] proved the existence of weak solutions of the non-stationary system in two and three space dimensions for a class of physical relevant and singular free energy densities with the same density. Desjardins [9] established regularity results for this system. For the incompressible Navier-Stokes-Allen-Cahn system with $\frac{\partial f}{\partial\chi}=\frac{1}{\delta}(\chi^3-\chi)$, Xu et al. [10] investigated a family of axisymmetric solutions for isentropic model in $\mathbb{R}^3$ and obtained global regularity of the constructed solutions with small initial data. Zhao et al. [11] studied the vanishing viscosity limit for isentropic model and proved the local existence of smooth solutions in a bounded domain. Li et al. [1,2] established the existence and uniqueness of the local strong solution and a blow-up criterion for two-phase flow of viscous incompressible fluids with different densities in a 2D or 3D bounded domain.

This paper is devoted to establishing the regularity criterion for the strong solutions of the 3D incompressible density-dependent Navier-Stokes-Allen-Cahn system. Without