JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 3, pp. 286-290

doi: 10.4208/jpde.v28.n3.7
September 2015

Remarks on Liouville Type Result for the 3D Hall-MHD System

ZHANG Zujin[1,*], YANG Xian[2] and QIU Shulin[1]

[1] *School of Mathematics and Computer Sciences, Gannan Normal University, Ganzhou 341000, China.*
[2] *Foreign Language Department, Ganzhou Teachers College, Ganzhou 341000, China.*

Received 17 May 2015; Accepted 31 July 2015

Abstract. In this paper, we consider the 3D Hall-MHD system, and provide an improved Liouville type result for its stationary version.

AMS Subject Classifications: 35Q35, 35B65

Chinese Library Classifications: O175.29, O175.28

Key Words: Hall-MHD system; Liouville theorem.

1 Introduction

This paper concerns itself with the three-dimensional (3D) Hall-magnetohydrodynamics system (Hall-MHD):

$$\begin{cases} \boldsymbol{u}_t+(\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla\pi=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\triangle\boldsymbol{u}, & \\ \nabla\cdot\boldsymbol{u}=0, & \text{in } \mathbb{R}^3\times(0,\infty), \\ \boldsymbol{B}_t-\nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\nabla\times[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]=\triangle\boldsymbol{B}, & \\ \boldsymbol{u}(0)=\boldsymbol{u}_0, \quad \boldsymbol{B}(0)=\boldsymbol{B}_0, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\boldsymbol{u}$ is fluid velocity field, $\boldsymbol{B}$ is the magnetic field, and π is a scalar pressure. We prescribe the initial data to satisfy the condition

$$\nabla\cdot\boldsymbol{u}_0=\nabla\cdot\boldsymbol{B}_0=0. \tag{1.2}$$

The first systematic study of the Hall-MHD system is pioneered by Lighthill [1] followed by Campos [2]. Comparing with the usual MHD equations, the Hall-MHD system

*Corresponding author. *Email addresses:* `zhangzujin361@163.com` (Z. J. Zhang), `yangxianxisu@163.com` (X. Yang), `qiushulin2003@163.com` (S. L. Qiu)

has the Hall term $\nabla\times[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]$ in $(1.1)_3$, which may become significant for such problems as magnetic reconnection in geo-dynamo [3], star formation [4,5], neutron stars [6] and space plasmas [7,8].

Mathematically, the Hall-MHD system can be derived from either two-fluids or kinetic models (see [3]), and the global existence of weak solutions, local existence and uniqueness of smooth solutions, blow-up criteria and small data global existence of classical solutions were established in [9, 10]. For the fractional Hall-MHD, the reader is referred to [11].

The stationary version of (1.1) is

$$\begin{cases} (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}+\nabla\pi=(\nabla\times\boldsymbol{B})\times\boldsymbol{B}+\triangle\boldsymbol{u}, \\ \nabla\cdot\boldsymbol{u}=0, \\ -\nabla\times(\boldsymbol{u}\times\boldsymbol{B})+\nabla\times[(\nabla\times\boldsymbol{B})\times\boldsymbol{B}]=\triangle\boldsymbol{B}. \end{cases} \tag{1.3}$$

And in [9], the authors established the following Liouville type theorem.

Theorem 1.1. ([9]) *Let $\boldsymbol{u},\boldsymbol{B}$ be $C^2(\mathbb{R}^3)$ solutions to* (1.3) *satisfying*

$$\boldsymbol{u},\boldsymbol{B}\in L^\infty(\mathbb{R}^3)\cap L^{\frac{9}{2}}(\mathbb{R}^3);\qquad \nabla\boldsymbol{u},\nabla\boldsymbol{B}\in L^2(\mathbb{R}^3).$$

Then we have $\boldsymbol{u}=\boldsymbol{B}=\boldsymbol{0}$.

It is not natural to assume that the boundedness of the solution $\boldsymbol{u},\boldsymbol{B}$ (see [12,13]), and the aim of this paper is to improving Theorem 1.1 as

Theorem 1.2. *Let $\boldsymbol{u},\boldsymbol{B}$ be $C^2(\mathbb{R}^3)$ solutions to* (1.3) *satisfying*

$$\boldsymbol{u},\boldsymbol{B}\in L^{\frac{9}{2}}(\mathbb{R}^3);\qquad \nabla\boldsymbol{u},\nabla\boldsymbol{B}\in L^2(\mathbb{R}^3).$$

Then we have $\boldsymbol{u}=\boldsymbol{B}=\boldsymbol{0}$.

2 Proof of Theorem 1.2

In this section, we shall prove Theorem 1.2.

We first derive an estimate of the pressure. Taking the divergence of $(1.3)_1$, and using the vector identity

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=-\nabla\frac{|\boldsymbol{B}|^2}{2}+(\boldsymbol{B}\cdot\nabla)\boldsymbol{B},$$

we obtain

$$-\triangle\pi=\sum_{i,j=1}^{3}\partial_i\partial_j(u_iu_j-B_iB_j)+\triangle\frac{|\boldsymbol{B}|^2}{2}.$$

Classical elliptic regularity results then yields

$$\|\pi\|_{L^q}\le C\|(\boldsymbol{u},\boldsymbol{B})\|_{L^{2q}}^2,\quad 1<q<\infty. \tag{2.1}$$