JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **28**, No. 1, pp. 9-22

doi: 10.4208/jpde.v28.n1.2
March 2015

Wave Kernels with Bi-Inverse Square Potentials on Euclidean Plane

MOUSTAPHA Mohamed Vall Ould*

Université des Sciences, de Technologie et de la Médecine (*USTM*), *Faculté des Sciences et Techniques.*
Unité de recherche: Analyse EDP et Modélisation, Facult des Sciences et Techniques, Universit des Sciences de Technologie et de la médecine, Nouakchott BP: 5026 Mauritanie.

Received 1 June 2014; Accepted 17 December 2014

Abstract. The Cauchy problem for the wave equation with bi-inverse square potential on Euclidean plane is solved in terms of the two variables Appell F_2 hypergeometric functions. Our principal tools are the Hankel transforms and the special functions of mathematical physics.

AMS Subject Classifications: 35J05; 35J10

Chinese Library Classifications: O175.25

Key Words: Inverse square potential; wave equation; Hankel transform; Bessl functions; F_2-Appell hypergeometric function.

1 Introduction and statement of results

Consider the following Cauchy problem of the wave type on Euclidean plane

$$\begin{cases} L_{(\nu,\nu')}u(t,p)=\partial_t^2 u(t,p), & (t,p)\in\mathbb{R}\times{\mathbb{R}_+^*}^2, \\ u(0,p)=0, \quad \partial_t u(0,p)=f(p)\in C_0^\infty({\mathbb{R}_+^*}^2), \end{cases} \tag{1.1}$$

where

$$L_{(\nu,\nu')}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{1/4-\nu^2}{x^2}+\frac{1/4-\nu'^2}{y^2}, \tag{1.2}$$

*Corresponding author. *Email addresses:* khames@ustm.mr (M. V. O. Moustapha)

is called here the Schrödinger operator with bi-inverse square potential and ν and ν' are real parameters. For the classical Schrödinger operator with inverse square potential

$$L_\nu = \frac{\partial^2}{\partial x^2} + \frac{1/4-\nu^2}{x^2}, \tag{1.3}$$

on the half real line $\mathbb{R}_+^*$, the solution of the problem (1.1) is (Taylor [1], p.132-133):

$$u(t,x) = \int_0^\infty W_\nu(\sigma(t,x,x')f(x')\mathrm{d}x', \tag{1.4}$$

where

$$W_\nu(\sigma) = \begin{cases} 0, & \text{when } 1<\sigma, \\ \frac{1}{2}P_{\nu-\frac{1}{2}}(\sigma), & \text{when } -1<\sigma<1, \\ \frac{\cos\pi\nu}{\pi}Q_{\nu-\frac{1}{2}}(-\sigma), & \text{when } \sigma<-1, \end{cases} \tag{1.5}$$

where P_m and Q_m denote the Legendre functions of degree m of the first and second kind respectively:

$$P_m(\sigma) = F(-m,m+1,1;(1-\sigma)/2),$$
$$Q_m(\sigma) = B(1/2,m+1)\frac{1}{(2\sigma)^{m+1}}F((m+1)/2,(m+2)/2;m+3/2;1/\sigma^2),$$

and $\sigma(t,x,x') = (x^2+x'^2-t^2)/(2xx')$.

The Gauss hypergeometric function is defined by:

$$F(a,b,c;z) = \sum_{n=0}^\infty \frac{(a)_n(b)_n}{(c)_n n!}z^n, \quad (|z|<1),$$

where as usual $(a)_n$ is the Pochhamer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{1.6}$$

The functions B and Γ are the classical Euler functions.

The inverse square potential arises in several contexts, one of them is the Schrödinger equation in non relativistic quantum mechanics (Reed and Simon [2]) . For example, the Hamiltonian for a spinzero particle in Coulomb field gives rise to a Schrödinger operator involving the inverse square potential (Case [3]).

The Cauchy problem for the wave equation with the inverse square potential in Euclidean space $\mathbb{R}^n$ is extensively studied (Burg et al [4]), (Cheeger and Taylor [5]), (Lamb [6]) and (Planchon et al [7]). The bi-inverse square potential has been considered by (Boyer [8]) in the case of the Schrödinger equation.