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Positive Solutions of Nonlinear Elliptic Problem in a Non-Smooth Planar Domain

BEN BOUBAKER Mohamed Amine*

Institut Preparatoire aux Etudes D'Ingénieur De Nabeul University of Carthage Campus Universitaire Merezka, 8000, Nabeul, Tunisia.

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Abstract. We find and prove 3G inequalities for the Laplacian Green function with the Dirichlet boundary condition, which are applied to show the existence of positive continuous solutions of the nonlinear equation

$$\Delta u - V u = g(\cdot,u),$$

where V and g are Borel measurable functions, required to satisfy suitable assumptions related to a new functional class J. Our approach uses the Schauder fixed point theorem.

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1 Introduction

In this paper, we work in the Euclidean space $\mathbb{R}^2$. By S_α, we denote the domain defined by

$$S_\alpha = \left\{ z \in \mathbb{C} : |\arg z| < \frac{\pi}{2\alpha} \right\}, \quad \alpha \in \left] \frac{1}{2}, +\infty \right[\setminus \{1\}.$$

The objective is to study the existence of continuous solutions for the nonlinear elliptic problem

$$\begin{cases} \Delta u - V u = g(\cdot,u), & \text{in } S_\alpha \text{ (in the sense of distributions)}, \\ u > 0, & \text{in } S_\alpha, \\ u = 0, & \text{on } \partial S_\alpha, \end{cases} \tag{1.1}$$

*Corresponding author. *Email address:* `MohamedAmine.BenBoubaker@ipein.rnu.tn` (M. A. Ben Boubaker)

where V and g are Borel measurable functions, required to satisfy suitable assumptions related to a new functional class J. The existence results of problem (1.1) have been extensively studied when $V=0$ and the special nonlinearity $g(x,t)=p(x)q(t)$, for both bounded and unbounded domain D in $\mathbb{R}^n (n\geq 1)$, with smooth compact boundary (see for example [1–4]). On the other hand, in [5–7], the authors considered the problem (1.1) where there is no restrictions on the sign of g, and D is an unbounded domain in $\mathbb{R}^n (n\geq 1)$ with a compact Lipschitz boundary. Then, they proved the existence of infinitely many solutions provided that g is in a certain Kato class. Namely, they showed that there exists a number $b_0>0$ such that for each $b\in(0,b_0]$, there exists a positive continuous solution u in $\overline{D}$ satisfying

$$\lim_{|x|\longrightarrow+\infty}\frac{u(x)}{h(x)}=b,$$

where h is a positive solution of the homogeneous Dirichlet problem $\Delta u=0$ in D, $u=0$ on ∂D. In [8], I. Bachar, L. Maatoug and H. Maagli studied (1.1) with $V=0$ in $D=\{(x_1,x_2)\in\mathbb{R}^2 : x_2>0\}$ and showed the existence of positive solutions with the growth as x_2 at infinity. In [9], K. Hirata studied (1.1) in uniform cone Γ in $\mathbb{R}^n$ $(n\geq 3)$. By applying sharp estimates for the Green function, he proved the existence of many continuous solutions with the growth as the Martin kernels at infinity. Our aim is to extend the result of [9] to S_α in $\mathbb{R}^2$. The paper is organized as follows. In Section 2, we establish some preliminary results that will be necessary throughout this paper. In Section 3, we first give an estimation of the Green function G_α of the Laplacian with Dirichlet boundary conditions on S_α, then we prove some inequalities on G_α. In particular, we establish a new version of the 3G-Theorem on S_α.

Theorem 1.1. (3G-Theorem) *There exists a constant $C_1=C_1(\alpha)>0$ such that for all x,y and z in S_α, we have*

$$\frac{G_\alpha(x,z)\,G_\alpha(z,y)}{G_\alpha(x,y)}\leq C_1\left(\left(\frac{|z|}{|x|}\right)^{(\alpha-1)}\frac{\delta(z)}{\delta(x)}G_\alpha(x,z)+\left(\frac{|z|}{|y|}\right)^{(\alpha-1)}\frac{\delta(z)}{\delta(y)}G_\alpha(y,z)\right), \tag{1.2}$$

where $\delta(x)$ is the Euclidean distance from x to ∂S_α.

This enables us to define and study, in Section 4 a new Kato class J of functions on S_α. Let

$$H_\alpha(x,y)=\left(\frac{|y|}{|x|}\right)^{(\alpha-1)}\frac{\delta(y)}{\delta(x)}G_\alpha(x,y),\quad \text{for } x,y\in S_\alpha.$$

Definition 1.1. *We say that a Borel measurable function q on S_α belongs to the Kato class J if q satisfies the following conditions:*

$$\lim_{r\longrightarrow 0}\sup_{x\in S_\alpha}\int_{(|x-y|\leq r)\cap S_\alpha}H_\alpha(x,y)\,|q(y)|\,\mathrm{d}y=0, \tag{1.3}$$

$$\lim_{M\longrightarrow+\infty}\sup_{x\in S_\alpha}\int_{(|y|\geq M)\cap S_\alpha}H_\alpha(x,y)\,|q(y)|\,\mathrm{d}y=0. \tag{1.4}$$