JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **27**, No. 1, pp. 64-73

doi: 10.4208/jpde.v27.n1.4
March 2014

BKM's Criterion of Weak Solutions for the 3D Boussinesq Equations

YANG Xinguang* and ZHANG Lingrui

College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China.
College of Education and Teacher Development, Henan Normal University, Xinxiang 453007, China.

Received 24 May 2013; Accepted 2 July 2013

Abstract. In this present paper, we investigate the Cauchy problem for 3D incompressible Boussinesq equations and establish the Beale-Kato-Majda regularity criterion of smooth solutions in terms of the velocity field in the homogeneous *BMO* space.

AMS Subject Classifications: 52B10, 65D18, 68U05, 68U07

Chinese Library Classifications: O175.27

Key Words: Incompressible Boussinesq equations; BKM's criterion; smooth solutions.

1 Introduction

This paper is devoted to establish BKM's criterion of smooth solutions for the Cauchy problem for 3D Boussinesq equations with viscosity in $\mathcal{R}^3$

$$u_t+u\cdot\nabla u-\eta\triangle u+\nabla p=\theta e_3, \tag{1.1}$$
$$\theta_t+u\cdot\nabla\theta-\nu\triangle\theta=0, \tag{1.2}$$
$$\nabla\cdot u=0, \tag{1.3}$$
$$t=0:\ u=u_0(x),\ \ \theta=\theta_0(x), \tag{1.4}$$

here u is the velocity field, p is the pressure, θ is the small temperature deviations which depends on the density. $\eta\geq 0$ is the viscosity, $\nu\geq 0$ is called the molecular diffusivity and $e_3=(0,0,1)^T$. The above systems describe the evolution of the velocity field u for a three-dimensional incompressible fluid moving under the gravity and the earth rotation which

*Corresponding author. *Email addresses:* yangxinguang@hotmail.com (X. Yang), zhanglingrui@126.com (L. Zhang)

come from atmospheric or oceanographic turbulence where rotation and stratification play an important role. When the initial density θ_0 is identically zero (or constant) and $\eta=0$, then (1.1)-(1.4) reduce to the classical incompressible Euler equation:

$$u_t+u\cdot\nabla u+\nabla p=0, \tag{1.5}$$
$$\nabla\cdot u=0, \tag{1.6}$$
$$u(x,t)|_{t=0}=u_0(x). \tag{1.7}$$

For the incompressible Euler equation and Navier-Stokes equation, a well-known criterion for the existence of global smooth solutions is the Beale-Kato-Majda criterion in [1] which states the control of the vorticity when $\omega=\mathrm{curl}\ u$ in $L^1(0,T;L^\infty)$, this is sufficient to get the global well-posedness of solutions, i.e., any solution u is smooth up to time T under the assumption that $\int_0^T\|\nabla\times u(t)\|_{L^\infty}\mathrm{d}t<+\infty$. Kozono and Taniuchi [2] improved the Beale-Kato-Majda criterion under the assumption $\int_0^T\|\nabla\times u(t)\|_{\mathrm{BMO}}\mathrm{d}t<+\infty$. The regularity criteria for the Navier-Stokes equations, we can refer to Bahouri, Chemin and Danchin [3], Cao and Titi [4], Kato and Ponce [5], Kozono and Taniuchi [2], Zhou [6,7], Zhou and Lei [8], Zhang and Chen [9].

The global well-posedness for two-dimensional Boussinesq equations which has recently drawn a lot of attention. More precisely, the global well-posedness has been shown in various function spaces and for different viscosity, we can refer to [10–20]. When $\eta=\nu=0$, the Boussinesq system exhibits vorticity intensification and the global well-posedness issue remains an unsolved challenging open problem (if θ_0 is a constant) which may be formally compared to the similar problem for the three-dimensional axisymmetric Euler equations with swirl.

For the three-dimensional case, Hmidi and Rousset [16,17] proved the global well-posedness for the 3D Navier-Stokes-Boussinesq equations and Euler-Boussinesq equations with axisymmetric initial data without swirl respectively. Danchin and Paicu [12] obtained the global existence and uniqueness result in Lorentz space for the Boussinesq equations with small data.

Our purpose of this paper is to obtain logarithmically improved regularity (BKM's) criterion of smooth solutions in terms of velocity field in BMO space.

Now we state our result as follows.

Theorem 1.1. *Assume that* $(u_0,\theta_0)\in H^m(\mathcal{R}^3)$ *holds with* $\mathrm{div}u_0=0$ *and* $m\geq 3$. *If* u *satisfies the condition*

$$\int_0^T\frac{\|\nabla\times u(t)\|_{\mathrm{BMO}}}{\sqrt{\ln(e+\|\nabla\times u(t)\|_{\mathrm{BMO}})}}\mathrm{d}t<+\infty, \tag{1.8}$$

then the solution (u,θ) *for the Cauchy problem* (1.1)-(1.4) *can be extended smoothly beyond* T.

The paper is organized as follows. We shall state some important inequalities in Section 2 and prove Theorem 1.1 in Section 3.