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A Pseudo-Parabolic Type Equation with Weakly Nonlinear Sources

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Abstract. In this paper, we prove the existence of nonnegative solutions to the initial boundary value problems for the pseudo-parabolic type equation with weakly nonlinear sources. Further, we discuss the asymptotic behavior of the solutions as the viscous coefficient k tends to zero.

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1 Introduction

This paper is concerned with the following initial boundary value problem for one spatial dimensional diffusion equation

$$\frac{\partial u}{\partial t}-k\frac{\partial D^2 u}{\partial t}=D^2u+u^q, \qquad (x,t)\in Q_T, \tag{1.1}$$

$$u(0,t)=u(1,t)=0, \qquad t\in[0,T], \tag{1.2}$$

$$u(x,0)=u_0(x), \qquad x\in[0,1], \tag{1.3}$$

where $Q_T=(0,1)\times(0,T)$, $T>0$ is a given constant, $0<q<1$, $D=\partial/\partial x$, $k>0$ denotes the viscous coefficient. The purpose of this paper is to investigate the solvability of the problem (1.1)-(1.3) and the asymptotic behavior of solutions as the viscous coefficient k tends to zero.

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Equations of the form (1.1) are called pseudo-parabolic equations or Sobolev type equations, and come from many interesting phenomena in fluid mechanics, mathematical biology, etc. For example, this type of equations can be used to model the uptake of liquids by polymers [1], fluid flow in fissured porous media [2], two phase flow in porous media with dynamical capillary pressure [3], heat conduction involving a thermodynamic temperature $\theta=u-k\Delta u$ and a conductive temperature u [4], flow of some non-Newtonian fluids [5], aggregation of populations [6], etc.

Mathematical study of pseudo-parabolic equations goes back to the works of Showalter & Ting in 1970s (see [7] and subsequent works). From then on, many interesting results about this type of equations have been obtained. For example, the existence and uniqueness of solutions [8, 9], the asymptotic behavior of solutions [10, 11] and the stability of the solutions [12]. In many cases, $k\Delta u_t$ can be seen as viscosity, which guarantees that the equation can be transformed into a nonlocal, Banach-space-value ordinary differential equation in time direction. Authors of [13, 14] considered the nonlinear diffusion equations with a pseudo-parabolic regularizing term being the Laplacian of the time derivative. Global existence of a strong solution is proved by writing the problem as a linear elliptic operator, acting on the time derivative, equal to the nonlinear diffusion term. In such situation, the linear elliptic operator, acting on the time derivative, could be inverted and then the standard geometric theory of nonlinear parabolic equations (see e.g. [15]) is applicable.

Though the viscous effect is common in the real world and is important for theoretical study, there are only a few research on the behavior of solutions as viscous coefficient tends to zero. In [16] and [17], the authors investigated the pseudo-parabolic type equation (1.1) with strongly nonlinear time periodic sources, i.e. $q>1$. In details, [16] applied the topological degree method and the energy method to obtain the existence and the asymptotic behavior of time periodic solutions. With the aid of comparison principle, by constructing a uniformly bounded supersolution and employing fixed point theorem, [17] proved the existence of unique solution to initial boundary value problems for small initial datum and discussed the limit process for $k\to 0$. In this paper, we are going to treat the case of $0<q<1$ for initial boundary value problems. We choose the approach of Leray-Schauder's theorem and approximation process, which is classical and effective. Actually, the third order term makes the proof complicated. Further, due to the weak nonlinear sources, we should first approximate u^q by $(u_\varepsilon+\varepsilon)^q$ with $\varepsilon\in(0,1)$ and discuss the solvability of the approximation problem by applying fixed point theorem. Basing on consistency estimates, we obtain the existence and the asymptotic behavior of solutions by approximation process.

The main results of this paper are as follows.

Theorem 1.1. *For any given initial datum* $u_0\in C^{2+\alpha}[0,1]$ *with* $u_0(0)=u_0(1)=0$, $u_0\geq 0$, $\alpha\in(0,1)$, *the problem* (1.1)-(1.3) *admits a nonnegative solution* $u\in C^{2+\alpha,1}(\overline{Q}_T)$ *satisfying* $D^2u_t\in C^{\alpha,0}(\overline{Q}_T)$.