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The Nehari Manifold and Application to a Quasilinear Elliptic Equation with Multiple Hardy-Type Terms

NYAMORADI N.*, SHEKARBIGI M., RAZLANSARI A.
and YAVARI M.

*Department of Mathematics, Faculty of Sciences, Razi University,
67149 Kermanshah, Iran.*

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Abstract. In this paper, by using the Nehari manifold and variational methods, we study the existence and multiplicity of positive solutions for a multi-singular quasilinear elliptic problem with critical growth terms in bounded domains. We prove that the equation has at least two positive solutions when the parameters λ belongs to a certain subset of $\mathbb{R}$.

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1 Introduction

In this paper, we consider the following quasilinear elliptic problem

$$\begin{cases} -\triangle_p u - \sum\limits_{i=1}^{k} \mu_i \dfrac{|u|^{p-2}u}{|x-a_i|^p} = f(x)|u|^{p^*-2}u + \lambda g(x)|u|^{q-2}u, & \text{in } \Omega, \\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^{\mathbb{N}}$ $(N \geq 3)$ is a bounded domain with smooth boundary $\partial\Omega$ such that the points $a_i \in \Omega$, $i=1,2,\cdots,k$, $k \geq 2$, $0 \leq \mu_i < \overline{\mu} := ((N-p)/p)^p$, and $p^* := (pN)/(N-p)$ is the critical Sobolev exponent and $1 \leq q < p$, $\lambda > 0$ and f,g are continuous functions.

*Corresponding author. *Email address:* `nyamoradi@razi.ac.ir` (N. Nyamoradi)

Problem (1.1) is related to the following Hardy inequality [1–3]:

$$\int_\Omega \frac{|u|^p}{|x-a|^p}\mathrm{d}x \le \frac{1}{\overline{\mu}}\int_\Omega |\nabla u|^p \mathrm{d}x, \quad \forall a\in\mathbb{R}^N, \quad u\in C_0^\infty(\mathbb{R}^N). \tag{1.2}$$

In this paper, for $\sum_{i=1}^k \mu_i\in[0,\overline{\mu})$, we use $W=W_0^{1,p}(\Omega)$ to denote the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$||u||=||u||_W=\left(\int_\Omega\left(|\nabla u|^p-\sum_{i=1}^k\mu_i\frac{|u|^p}{|x-a_i|^p}\right)\mathrm{d}x\right)^{\frac{1}{p}},$$

which by (1.2), this norm is equivalent to the standard norm $\left(\int_\Omega|\nabla u|^p\mathrm{d}x\right)^{\frac{1}{p}}$ on W.

Definition 1.1. *We say that $u\in W$ is weak solution to* (1.1) *if for all $v\in W$ we have*

$$\int_\Omega\left(|\nabla u|^{p-2}\nabla u\nabla v-\sum_{i=1}^k\mu_i\frac{|u|^{p-2}}{|x-a_i|^p}uv-f(x)|u|^{p^*-2}uv-\lambda g(x)|u|^{q-2}uv\right)\mathrm{d}x=0.$$

By the standard elliptic regularity argument, we have that the solution $u\in C^2(\Omega\setminus\{a_1,a_2,\cdots,a_k\})\cap C^1(\overline{\Omega}\setminus\{a_1,a_2,\cdots,a_k\})$. It is well known that the nontrivial solution of problem (1.1) is equivalent to the corresponding nonzero critical points of the energy functional

$$J_\lambda(u)=\frac{1}{p}\int_\Omega\left(|\nabla u|^p-\sum_{i=1}^k\mu_i\frac{|u|^p}{|x-a_i|^p}\right)\mathrm{d}x-\frac{1}{p^*}\int_\Omega f(x)|u|^{p^*}\mathrm{d}x-\frac{\lambda}{q}\int_\Omega g(x)|u|^q\mathrm{d}x,$$

for every $u\in W$.

For $0\le\mu_i<\overline{\mu}$ and $a_i\in\Omega$, $i=1,2,\cdots,k$, we let S_{μ_i} be the best Sobolev embedding constant defined by

$$S_{\mu_i}:=\inf_{u\in W\setminus\{0\}}\frac{\int_\Omega\left(|\nabla u|^p-\mu_i\frac{|u|^p}{|x-a_i|^p}\right)\mathrm{d}x}{\left(\int_\Omega|u|^{p^*}\mathrm{d}x\right)^{\frac{p}{p^*}}} \tag{1.3}$$

and from [4], we get that S_{μ_i} is independent of Ω.

In [5], the authors studied the following limiting problem:

$$\begin{cases}-\triangle_p u-\mu\frac{u^{p-1}}{|x-a_i|^p}=u^{p^*-1}, & \text{in } \mathbb{R}^N\setminus\{a_i\},\\ u>0,\ \ u\in D^{1,p}(\mathbb{R}^N), & \text{in } \mathbb{R}^N\setminus\{a_i\},\end{cases} \tag{1.4}$$

where $0\le\mu<\overline{\mu}$, $1<p<N$ and $D^{1,p}(\mathbb{R}^N)=\{u\in L^{p^*}(\mathbb{R}^N):\nabla u\in L^p(\mathbb{R}^N)\}$. They have prove that the problem (1.4) has radially symmetric ground state

$$V_{p,\mu_i,\varepsilon}^{a_i}(x)=\varepsilon^{\frac{p-N}{p}}U_{p,\mu_i}\left(\frac{x-a_i}{\varepsilon}\right)=\varepsilon^{\frac{p-N}{p}}U_{p,\mu_i}\left(\frac{|x-a_i|}{\varepsilon}\right), \quad \forall\varepsilon>0,$$