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# On Approximation and Computation of Navier-Stokes Flow

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**Abstract.** We present an approximation method for the non-stationary nonlinear incompressible Navier-Stokes equations in a cylindrical domain $(0,T)\times G$, where $G\subset\mathbb{R}^3$ is a smoothly bounded domain. Our method is applicable to general three-dimensional flow without any symmetry restrictions and relies on existence, uniqueness and representation results from mathematical fluid dynamics. After a suitable time delay in the nonlinear convective term $v\cdot\nabla v$ we obtain globally (in time) uniquely solvable equations, which – by using semi-implicit time differences – can be transformed into a finite number of Stokes-type boundary value problems. For the latter a boundary element method based on a corresponding hydrodynamical potential theory is carried out. The method is reported in short outlines ranging from approximation theory up to numerical test calculations.

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**Key Words**: Navier-Stokes equations; regularization; time delay; finite differences; Stokes resolvent; hydrodynamical potential theory; boundary element methods; numerical simulation.

## 1 Introduction

The Navier-Stokes equations represent the basic model for the motion of a viscous incompressible fluid that coincides with the principles of momentum and mass conservation. Despite their simplicity, theses equations seem to govern the motion of air, water and many other fluids very accurately over a wide range of conditions. Even the motion of blood in large and medium sized arteries is considered as a homogeneous (Newtonian) fluid flow and can be described by the Navier-Stokes equations [1]. Thus, their mathematical theory is central to the rigorous analysis of many experimental observations.

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During the last 80 years, a great deal of progress has been made on both the basic mathematical theory of the equations and on its application to the understanding of physical phenomena. Nevertheless, one of the most important matters, that of estimating the regularity of solutions over long periods of time, remains unsolved and a fascinating challenge up to now.

To present the problem, let $T>0$ be a given real number and $G\subset\mathbb{R}^3$ a bounded domain with smooth boundary $\partial G$. In $G$ we consider a non-stationary viscous incompressible fluid flow and assume that it can be described by the Navier-Stokes equations

$$\begin{cases} \partial_t v-\nu\Delta v+v\cdot\nabla v+\nabla p=F, \\ \nabla\cdot v=0, \\ v|_{\partial G}=0, \\ v|_{t=0}=v_0, \end{cases} \qquad (t,x)\in(0,T)\times G. \tag{1.1}$$

Here $v(t,x)=(v_1(t,x),v_2(t,x),v_3(t,x))$ represents the velocity vector of a fluid particle at time $t\in(0,T)$ and at location $x\in G$, and accordingly $p(t,x)$ is a scalar (kinematic) pressure function. The vector function $F$ is the external force density and $\nu$ a positive material constant, the so-called kinematic viscosity. We denote by $\partial_t v$ the partial derivative of $v$ with respect to time, $\Delta$ is the Laplace operator and $\nabla$ the gradient in $\mathbb{R}^3$. The term $v\cdot\nabla v:=(v\cdot\nabla)v$ is the nonlinear convective term that makes these equations so difficult. It results from the material derivative of the velocity field, and it is defined by applying the formal scalar product of $v$ and $\nabla$ to each component of the velocity $v$. From the physical point of view, the first three equations describe the equilibrium of forces, and the fourth equation – $\mathrm{div}\, v=\nabla\cdot v=0$ – expresses the conservation of masses, i.e., the fact that the fluid flow is incompressible.

Furthermore, there are boundary and initial conditions: We assume that the fluid adheres to the boundary, i.e., the so-called no-slip condition, and that the velocity at initial time $t=0$ is given by a prescribed initial velocity distribution $v_0$.

These are the equations of Navier-Stokes, perhaps the most important system of partial differential equations and one of seven famous Mathematical Millennium Prize Problems (see `http://www.claymath.org/millennium/`). The equations represent a system of four nonlinear partial differential equations concerning four unknown functions: Three components of the velocity field $v$ and a pressure function $p$. In the following we consider these equations always on a fixed given cylindrical domain $(0,T)\times G$, where $G\subset\mathbb{R}^3$ is smoothly bounded.

The problem of finding a uniquely determined classical solution defined for all times $t\in(0,T)$, if smooth data without further restrictions are given, has up to now only been solved for the case of two-dimensional flows, i.e., if $G$ is a domain in $\mathbb{R}^2$ and hence all functions involved depend on two spatial variables only [2]. Solving the general three-dimensional case presents difficulties which up to now have not been overcome [3–9]. Regarding numerical methods, the complexity of the situation is similar. The present paper suggests a method for the numerical solution of the equations of Navier-Stokes