JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **26**, No. 2, pp. 122-137

doi: 10.4208/jpde.v26.n2.3
June 2013

Implementation of Some Methods of Shape Design for Variational Inequalities

BUTT Rizwan*

Mathematics Department, College of Science, King Saud University, P. O. Box 2455, Riyadh 11451, Saudi Arabia.

Received 19 April 2012; Accepted 10 January 2013

Abstract. In this paper we present some results concerning the optimal shape design problem governed by the fourth-order variational inequalities. The problem can be considered as a model example for the design of the shapes for elastic-plastic problem. The computations are done by finite element method, and the performance criterion is minimized by the material derivative method. We also discuss the error estimates in the appropriate norm and present some numerical results. An example is used to clearly illustrate the essential elements of shape design problems.

AMS Subject Classifications: 35-XX, 35Bxx, 35Qxx

Chinese Library Classifications: O175.2

Key Words: Sobolev space; convexity; optimal shape design; variational inequalities; approximation; optimization techniques; finite element method; material derivative method.

1 Introduction

The purpose of this paper is to develop an optimal shape design which governed by the variational inequalities of the fourth-order for the design of the shape for an elastic-plastic problem. Much work has been done in the optimal shape design for systems described by partial differential equations [1], the main idea of this paper is to obtain the optimal shape design for the systems described by differential inequalities by introducing penalized differential equations and then taking limits of the equations resulting from the penalized or differential approximation. We develop the material derivative method [2] for the optimal shape design of an elastic-plastic problem. The problem arises when studying the torsion of a cylindrical bar of the section $\Omega\subset\Re^2$, the bar is made of an elastic plastic material, f is a constant proportional to the angle of twist of the end section of the bar which is not clamped. Several authors have proposed and studied the elastic plastic

*Corresponding author. *Email address:* `fouzanbutt@yahoo.com` (R. Butt)

problem (see, [3–5]). For the formulation of the problem we consider a domain Ω, which consists of two regions Ω_e and Ω_p (the elastic region and plastic region, respectively). Then the stress potential φ satisfies the fourth-order partial differential equation on Ω,

$$\mu\varphi + A\varphi = f \qquad \text{in } \Omega_e (\text{situated in the interior of } \Omega), \tag{1.1a}$$

$$|\Delta\varphi| = 1 \qquad \text{in } \Omega_p, \tag{1.1b}$$

with boundary conditions

$$\varphi = \partial\varphi/\partial n = 0 \quad \text{on } \Gamma, \tag{1.2}$$

where μ is a nonnegative constant.

Consider the Sobolev space $H^m(\Omega)$ of real-valued function having derivatives upto the order m, summed with square; $H_0^m(\Omega)$ be the closure of infinitely differentiable functions in the norm $H^m(\Omega)$. Let $V = H^2(\Omega) \cap H_0^2(\Omega)$, and

$$H_0^2(\Omega) = \left\{ \phi \middle| \phi \in H^2(\Omega),\ \phi|_\Gamma = \frac{\partial\phi}{\partial n}\Big|_\Gamma = 0 \right\}, \tag{1.3}$$

since the domain Ω is bounded and Γ is sufficiently regular the mapping

$$\phi \to \|\Delta\phi\|_{L^2(\Omega)},$$

defines on V a norm which is equivalent to that induced by $H^2(\Omega)$. Let us assume that $f \in H^{-2}(\Omega)$ (which is a topological dual of $H_0^2(\Omega)$); it is well known that (1.1a) admits one and only one solution in $H_0^2(\Omega)$; this solution is also the unique solution of the variational equation (of order 4), for all $\varphi \in H_0^2(\Omega)$

$$\int_\Omega (\Delta\varphi\Delta\phi + \mu\varphi\phi)dx = \langle f, \phi \rangle, \qquad \forall \phi \in H_0^2(\Omega), \tag{1.4}$$

which is also the solution of the minimization problem

$$\min_{\phi \in H_0^2(\Omega)} \left[1/2 \int_\Omega |\Delta\phi|^2 dx - \langle f, \phi \rangle \right]. \tag{1.5}$$

In (1.4) and (1.5), $\langle \cdot, \cdot \rangle$ represents the bilinear form of the duality between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$, and we thus have

$$\langle f, \phi \rangle = \int_\Omega f\phi dx,$$

and

$$a(\phi, \psi) = \int_\Omega \Delta\phi(x)\Delta\psi(x)dx, \quad \forall \phi, \psi \in H_0^2(\Omega).$$

Finally, let K be the closed convex subset of V, defined by

$$K = \{ \phi | \phi \in H_0^2(\Omega),\ |\Delta\phi| \le 1 \text{ a.e. in } \Omega \}. \tag{1.6}$$