JOURNAL OF PARTIAL DIFFERENTIAL EQUATIONS
J. Part. Diff. Eq., Vol. **26**, No. 1, pp. 39-47

doi: 10.4208/jpde.v26.n1.4
March 2013

On Conditions of the Nonexistence of Solutions of Nonlinear Equations with Data from Classes Close to L^1

KOVALEVSKY A. A.[1,*] and NICOLOSI F.[2]

[1] *Institute of Applied Mathematics and Mechanics, National Academy of Sciences of Ukraine, R. Luxemburg St. 74, 83114 Donetsk, Ukraine.*
[2] *Department of Mathematics and Informatics, University of Catania, Viale A. Doria 6, 95125 Catania, Italy.*

Received 14 June 2012; Accepted 2 January 2013

Abstract. We establish conditions of the nonexistence of weak solutions of the Dirichlet problem for nonlinear elliptic equations of arbitrary even order with some right-hand sides from L^m where $m>1$. The conditions include the requirement of a certain closeness of the parameter m to 1.

AMS Subject Classifications: 35J25, 35J40, 35J60

Chinese Library Classifications: O175.8, O175.25, O175.29

Key Words: Nonlinear elliptic equations in divergence form; Dirichlet problem; weak solution; existence and nonexistence of weak solutions.

1 Introduction

In the well-known work [1], a theory of entropy solutions for nonlinear elliptic second-order equations with L^1-data was developed. According to the results of this work, if Ω is a bounded open set of $\mathbb{R}^n$ $(n\geqslant 2)$, $1<p<n$, and coefficients of the equations under consideration grow with respect to the gradient of unknown function u as $|\nabla u|^{p-1}$ and satisfy natural coercivity and strict monotonicity conditions, then the Dirichlet problem in Ω for these equations has a unique entropy solution for every L^1-right-hand side. In addition, if $p>2-1/n$, the entropy solution is a weak solution. At the same time in [1] it was shown that if $1<p\leqslant 2-1/n$, the Dirichlet problem for the equation $-\Delta_p u+u=f$ in Ω does not have weak solutions for some $f\in L^1(\Omega)$.

*Corresponding author. *Email addresses:* alexkvl@iamm.ac.donetsk.ua (A. A. Kovalevsky), fnicolosi@dmi.unict.it (F. Nicolosi)

In connection with the above, now we note the following two cases where the Dirichlet problem for equations of the class under consideration has a weak solution for every right-hand side in $L^m(\Omega)$ (see in [2, Theorems 1.5.5 and 1.5.6]):

(a) $p \geqslant 2-1/n$, $m>1$;

(b) $p<2-1/n$, $m \geqslant n/(np-n+1)$.

In the present article, we give two nonexistence results. The first one concerns the above-mentioned Dirichlet problem for second-order equations. We prove that if $p< 2-1/n$ and $1<m<n/(np-n+1)$, then for some $f\in L^m(\Omega)$ the problem with the datum f does not have weak solutions (see Theorem 3.3). The second result concerns the Dirichlet problem in the same open set Ω for a class of $2l$-order equations whose coefficients admit the growth of rate $p-1>0$ with respect to the derivatives of order l of unknown function. We establish that under the conditions $n>2$, $2\leqslant l<n$, $p<2-l/n$ and $1<m<n/(np-n+l)$ for some $f\in L^m(\Omega)$ the problem with the datum f does not have weak solutions (see Theorem 4.1). We remark that the proof of these results given in Sections 3 and 4 respectively is based on the use of an assertion which establishes a relation between the parameters n, l, p and m of an operator acting from $L^m(\Omega)$ into $(\overset{\circ}{W}{}^{l,p}(\Omega))^*$ (see Proposition 2.3 in Section 2).

We note that a condition of the nonexistence of weak solutions of the Dirichlet problem for high-order equations with L^1-data was established in the recent article [3].

As far as the solvability of nonlinear elliptic high-order equations with L^1-right-hand sides is concerned, to our knowledge, there are no results on this subject in the general case. Some results on the existence of entropy and weak solutions of the Dirichlet problem for nonlinear elliptic high-order equations with coefficients satisfying a strengthened coercivity condition and L^1-data were obtained for instance in [4] and [5]. In this connection see also [2, Chapter 2] where a theory of the existence and properties of entropy and weak solutions of the Dirichlet problem for nonlinear fourth-order equations with strengthened coercivity and data from L^1 and classes close to L^1 is presented.

2 Auxiliary assertions

Let $n\in\mathbb{N}$, $n\geqslant 2$, and let Ω be a bounded open set of $\mathbb{R}^n$.

Propositon 2.1. *Let $m>1$, $l\in\mathbb{N}$, $p>1$, and let $H: L^m(\Omega)\to(\overset{\circ}{W}{}^{l,p}(\Omega))^*$ be an operator such that*

$$f\in L^m(\Omega), \varphi\in C_0^\infty(\Omega) \Longrightarrow \langle Hf,\varphi\rangle = \int_\Omega f\varphi \mathrm{d}x. \tag{2.1}$$

Then $\overset{\circ}{W}{}^{l,p}(\Omega)\subset L^{m/(m-1)}(\Omega)$.

Proof. First of all we observe that the operator H is linear. In fact, let $f,g\in L^m(\Omega)$ and $\alpha,\beta\in\mathbb{R}$. We fix an arbitrary function $\varphi\in\overset{\circ}{W}{}^{l,p}(\Omega)$ and a sequence $\{\varphi_k\}\subset C_0^\infty(\Omega)$ such that