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# Global Estimates in Orlicz Spaces for $p$-Laplacian Systems in $\mathbb{R}^N$

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**Abstract.** In this paper we obtain global estimates in Orlicz spaces for weak solutions of $p$-Laplacian systems in $\mathbb{R}^N$ for $N \geq 2$. Our results improve the known results for such problems.

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## 1 Introduction

Assume that $m \in \mathbb{N}$, $N \geq 2$ and $1 < p < N$ are fixed. Denote $p^* = Np/(N-p)$ the Sobolev conjugate of $p$. We study the following quasilinear elliptic system of $p$-Laplacian type

$$\operatorname{div}\left(|Du|^{p-2}Du\right) = \operatorname{div}(|\mathbf{F}|^{p-2}\mathbf{F}), \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $\mathbf{F} = (F^1, F^2, \cdots, F^m) \in L^p(\mathbb{R}^N, \mathbb{R}^{Nm})$ is a given $N \times m$ matrix.

As usual, the solutions of (1.1) are taken in a weak sense. We now state the definition of weak solutions.

**Definition 1.1.** *A function $u = (u^1, u^2, \cdots, u^m) \in D^p(\mathbb{R}^N, \mathbb{R}^m)$ is a weak solution of* (1.1) *if for any $\psi = (\psi^1, \psi^2, \cdots, \psi^m) \in C_0^\infty(\mathbb{R}^N, \mathbb{R}^m)$, we have*

$$\int_{\mathbb{R}^N} |Du|^{p-2}\langle Du, D\psi\rangle \, dx = \int_{\mathbb{R}^N} |\mathbf{F}|^{p-2}\langle \mathbf{F}, D\psi\rangle dx,$$

*where*

$$D^p(\mathbb{R}^N, \mathbb{R}^m) = \left\{ u \in L^{p^*}(\mathbb{R}^N, \mathbb{R}^m) \mid Du \in L^p(\mathbb{R}^N, \mathbb{R}^{Nm}) \right\}.$$

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It is known that when $\mathbf{F}\in L^p(\mathbb{R}^N,\mathbb{R}^{Nm})$, there is a unique weak solution $u\in D^p(\mathbb{R}^N,\mathbb{R}^m)$ of (1.1) satisfying

$$\int_{\mathbb{R}^N}|Du|^p\mathrm{d}x\le C\int_{\mathbb{R}^N}|\mathbf{F}|^p\mathrm{d}x. \tag{1.2}$$

Moreover, DiBenedetto and Manfredi [14] have obtained the following estimate

$$\int_{\mathbb{R}^N}|Du|^q\mathrm{d}x\le C\int_{\mathbb{R}^N}|\mathbf{F}|^q\mathrm{d}x \tag{1.3}$$

provided $\mathbf{F}\in L^q(\mathbb{R}^N,\mathbb{R}^{Nm})$ for any $q\ge p$.

In fact, there have been a wide research activities on the study on local/global $L^q$ estimates for the gradients of the general quasilinear elliptic system of $p$-Laplacian type with variable coefficients where the domain is a bounded domain in $\mathbb{R}^n$ (See [1–4] respectively).

Our interest is to study how the regularity of weak solutions is reflected by $\mathbf{F}$ in the setting of Orlicz spaces. In this work we are interested in extending the classical $L^q$, $q\ge p$, gradient estimates for weak solutions of (1.1) into the following estimates in Orlicz spaces

$$\int_{\mathbb{R}^N}\phi(|Du|^p)\,\mathrm{d}x\le C\int_{\mathbb{R}^N}\phi(|\,\mathbf{F}|^p)\,\mathrm{d}x, \tag{1.4}$$

where $\phi$ satisfies the global $\Delta_2\cap\nabla_2$ condition (see Definition 1.3). We remark that when $\phi(x)=|x|^{q/p}$ for $q\ge p$, (1.4) is reduced to $L^q$-estimate (1.3).

Orlicz spaces have been extensively studied in the area of analysis as one of the most natural generalizations of Sobolev spaces since they were introduced by Orlicz [5] (see [6–12]). The theory of Orlicz spaces plays a crucial role in many fields of mathematics including geometric, probability, stochastic, Fourier analysis and partial differential equations (see [13]).

Here for the reader's convenience, we will give some definitions on the general Orlicz spaces. We denote by $\Phi$ the function class that consists of all functions $\phi:[0,+\infty)\longrightarrow[0,+\infty)$, which are increasing and convex.

**Definition 1.2.** *A function $\phi\in\Phi$ is said to be a Young function if*

$$\lim_{t\to 0+}\frac{\phi(t)}{t}=\lim_{t\to+\infty}\frac{t}{\phi(t)}=0.$$

**Definition 1.3.** *A Young function $\phi$ is said to satisfy the global $\Delta_2$ condition, denoted by $\phi\in\Delta_2$, if there exists a positive constant $K$ such that for every $t>0$,*

$$\phi(2t)\le K\phi(t).$$

*Moreover, a Young function $\phi$ is said to satisfy the global $\nabla_2$ condition, denoted by $\phi\in\nabla_2$, if there exists a number $a>1$ such that for every $t>0$,*

$$\phi(t)\le\frac{\phi(at)}{2a}.$$